Review of Basic Bond Valuation

Working with fixed income instruments demands familiarity with the basics of bond math. These notes reviews bond valuation techniques.

I will assume you are familiar with:

A. How to value a zero coupon default-free and defaultable bond.
B. How to value a bond that pays a coupon, and compute the yield on a bond given market price and coupon.
C. How changes in market interest rates (yields) affect bond prices.
D. What is duration, and how it relates to (C).

We begin with the simplest possible questions – present value and future value. Take a single cash flow and ask,

1. What will it grow to in the future?
2. What is the [present] value of a single cash flow to be received in the future?

First, let us introduce some notation that we will use in the rest of the course,

\( PV_0 \) = present value today of a cash flow or a stream of cash flows (0 refers to time 0)
\( FV_t \) = future value at the end of time period \( t \)
\( i \) = nominal interest rate in annualized terms – i.e. per year.
\( t \) = number of time periods in years
\( d \) = maturity of security in days
\( F \) = face amount of security or investment
\( m \) = number of compounding periods per year
\( y \) = yield to maturity
\( P \) = price of security (today)

Future Value

Suppose we invest \( F \) dollars today for \( t \) time periods at an interest rate of \( i \) per year, then the amount of dollars at the end of time period \( t \) is the future value of the \( F \) dollars,

\[
FV_t = F (1 + i)^t
\]

Example:

Suppose you decide to invest $1 million in a financial security that pays 5% per year for the next 5 years.
\[ FV_5 = \$1 \text{ million} \times (1.05)^5 = \$1,276,281 \]

Typically in the money market sector, we will be dealing with time periods of less than a year – i.e. \( t < 1 \). For the case where \( t = 1 \), we can rewrite the future value formula as,

\[ i = \frac{FV_1 - F}{F} \]

This equation gives \( i \) as the annual return expressed as a simple interest rate – simple interest because it involves no compounding.

The general structure of a simple interest calculation is readily seen from this formula,

Annual return as simple interest = interest earned at year end/ principal invested.

What if \( t < 1 \), or the number of days to maturity \( d < 365 \). In this case we need an annualizing factor to convert the rate. For the case of a 365 day year and a day count convention (to be discussed later) of ACT/ACT, the formula is,

Annual return as simple interest = interest earned at year end/ principal invested \( \times \frac{365}{d} \)

**Compounding**

Often the security one is interested in generates a cash flow only once a year, but has a shorter compounding period so that interest is compounded multiple times a year.

**Example**

*Suppose you purchase a 3 month deposit that pays 5% simple interest, and then roll the proceeds into another 3 month deposit at 5% ... 4 times for an investment period of 1 year. Then the effective simple interest on this one year deposit would be,*

\[ i = (1 + 0.05/4)^4 - 1 = 5.095\% \]

*Note that the simple one year return is higher than 5%. This is, of course, the result of earning interest on interest – i.e. compounding.*

**Present Value**

In all of the above we calculated how an amount of money today would grow to an amount at time \( t \) – that is we calculated future value. Present value is just the reverse of this – given an amount of money at time \( t \), how much is it worth today.
\[ PV_0 = FV_t (1 + i)^{-t} \]

*Example*

*With rising education costs, you will need to set aside $250,000 in 25 years for your child’s college education. How much should you set aside today to fund this, if the interest rate is 5% per year?*

\[ PV_0 = \frac{250,000}{(1.05)^{25}} = 73,826 \]

A couple of simple properties of the present value formula are useful to keep in mind,

1. Present value decreases as the number of periods increases (\( t \) increases)
2. Present value decreases as the interest rate increases (\( i \) increases)

*Basic Bond Valuation*

The tools we have developed can be used to value bonds. Let us begin with a simple example.

*Example*

*The US Treasury auctions off 2 year notes at the end of each month. The notes pay coupons every 6 months, with the yield calculated on a semi-annual bond basis. We wish to compute the price of $1mm face a current 2 year note with coupon of 6% when the market [treasury] interest rate for 2 years is 5% (also on a semi-annual bond basis).

A semi-annual coupon of 6% on a $1mm bond translates to coupon payments of \( \$1mm \times 0.06/2 = \$30,000 \)

Then the value of the bond is,

\[ PV_0 = 30,000 \times (1.025^{-1} + 1.025^{-2} + 1.025^{-3} + 1.025^{-4}) + 1,000,000 \times 1.025^{-4} = 1,018,810 \]

or a bond price (per $100 face of bond) of,

\[ P = 101.881 \]

As an exercise, redo this calculation using a coupon of 5% and note that \( P = 100 \) exactly.
Yield to Maturity

Where does one arrive at a market treasury rate for 2 years of 5%? In practice, what we see when we look at the newspaper are bond prices, coupons and maturities.

For instance the following is a list of US Treasury bonds for different maturities,

<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 year</td>
<td>4.625</td>
<td>100-11</td>
</tr>
<tr>
<td>3 year</td>
<td>6.250</td>
<td>104-20</td>
</tr>
<tr>
<td>5 year</td>
<td>4.250</td>
<td>99-08</td>
</tr>
</tbody>
</table>

From this information, one can compute yields to maturity – the interest rate that sets the bond price equal to the present value of the cash flows.

Example

Let us calculate the yield to maturity on the 2 year bond. First note that the price is in units of 100, so 100 is the right face to use in our computations. A 4.625% semi-bond coupon on a 100 face means coupon flows of,

\[ 100 \times 0.04625/2 = 2.3125 \]

Then the price of a bond at an interest rate of \( y \) is,

\[ P = 2.3125 X (y^{-1} + y^{-2} + y^{-3} + y^{-4}) + 100 X y^{-4} \]

The market price of this bond is 100-11. In bond speak that means 100 + 11/32 - bond prices are quoted in fractions of 32nds.

\[ 100 + 11/32 = 2.3125 X (y^{-1} + y^{-2} + y^{-3} + y^{-4}) + 100 X y^{-4} \]

We can solve this equation for \( y \) to find the yield to maturity. 4.44% is the answer in this case. Thus in pricing other Treasuries or comparable securities of around 2 years in maturity the appropriate interest rate to use is 4.44%. 

Default Risk

The examples above consider the case of default free bond, which we usually take to be a US Treasury bond. Consider next the case of a corporate bond, which may default.

Example

Consider $1mm face value of a one-year corporate bond. The bond pays a coupon upon maturity of 5%. However, there is a 5% chance that the bond will default over the next year. In the event of default, investors only receive 50% of the face value of the bond (and no coupon). Suppose that the incidence of default is idiosyncratic and uncorrelated with the market, so that default is fully diversifiable. Assume that the market [US Treasury] interest rate for 1 year is 3%.

The cash-flows from this bond are as follows:
   a. If there is no default (probability 95%) the cash-flow is $1,050,000
   b. If there is default (probability 5%) the cash-flow is $500,000

The present value of the bond is therefore:

\[ PV_0 = 0.95 \times \frac{1,050,000}{1.03} + 0.05 \times \frac{500,000}{1.03} = 992,718 \]

Note that I have discounted by the riskless rate of 3% because of my assumption that the risk is fully diversifiable. If it was not, then I would have discounted by 3% plus the betaXrisk-premium, to reflect the correlation of the cash-flow with the market, times the market risk premium,

We can also compute the yield on this corporate bond. The yield on the bond is defined to be the return an investor will receive IN CASE OF NO DEFAULT. This is the definition of yield on a defaultable bond.

\[ y = \frac{1,050,000}{992,718} - 1 = 5.77\% \]

The corporate bond spread of 2.77% (=5.77 – 3) is compensation for the risk of default. To give a rough intuition of what drives this spread, consider the computation: there is a 5% chance that the investor will lose 50% of her money, and thus the investor demands around a 5%X50% =2.5% spread on the bond.

Duration

Duration is a commonly used measure for two reasons

1. It is a measure of the sensitivity of bond price with respect to changes in market interest rates.
2. It is a measure of the maturity of a bond – which can be useful for bonds that have irregular cash flows.
The key property of duration is that bonds with longer maturities have higher duration. This means that the prices of longer maturity bonds are more sensitive to changes in market interest rates.

For the purposes of this course, the sensitivity of prices will be the important use of duration. In fact, we will use a more convenient version of this, called the DV01.

Consider the following table of bond prices and yields of the 2 year 4.625 coupon that we worked with in the last example

<table>
<thead>
<tr>
<th>Yield</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.8</td>
<td>99.670</td>
</tr>
<tr>
<td>4.75</td>
<td>99.764</td>
</tr>
<tr>
<td>4.7</td>
<td>99.858</td>
</tr>
<tr>
<td>4.65</td>
<td>99.953</td>
</tr>
<tr>
<td>4.625</td>
<td>100.00</td>
</tr>
<tr>
<td>4.6</td>
<td>100.047</td>
</tr>
<tr>
<td>4.55</td>
<td>100.142</td>
</tr>
<tr>
<td>4.5</td>
<td>100.237</td>
</tr>
<tr>
<td>4.45</td>
<td>100.331</td>
</tr>
<tr>
<td>4.4</td>
<td>100.426</td>
</tr>
</tbody>
</table>

Suppose you purchased $1mm of this bond at issue for a price of 100, and immediately the fed cut interest rates so that 2 year yields fell by 12.5bps. How much profit would you make?

Well, if 2 year rates fell to 4.5%, the 4.625% coupon bond would clearly rise in price – from the table we can see that it would rise to 100.237, so that you would make

\[ \frac{1mm \times 0.237}{100} = \$2370 \]

The DV01 is defined as the profit or loss arising from a 1 bp change in the quoted interest rate of a $1mm face of a par bond. To compute the DV01 of the 2 year bond, we simply evaluate the bond price at 4.635%, resulting in a price of 99.981. Then,

\[ \text{DV01} = \frac{1mm \times (100 - 99.981)}{100} = \$189 \]

The number $189 is closely relate to the duration and maturity of the bond. As a rule of thumb, a T-year bond will have a DV01 of around T x 100. So for 2 year bond, DV01 \(\approx 200\). So the market price of a greater maturity bond is more sensitive to changes in interest rates.

Why is DV01 useful? Suppose that I am a trader and I just purchased the 2 year bond at par (100), and immediately the fed cut rates so that yields fell by 12.5bps, how much money did I make? Answer:

\[ 12.5 \text{ bps} \times \$189 = \$2363 \]