MORAL HAZARD AND THE OPTIMALITY OF DEBT

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ABSTRACT. I show that, in a benchmark model, debt securities minimize the welfare losses associated with the moral hazards of excessive risk-taking and lax effort. For any security design, the variance of the security payoff is a statistic that summarizes these welfare losses. Debt securities have the least variance, among all limited liability securities with the same expected value. In other models, mixtures of debt and equity are exactly optimal, and pure debt securities are approximately optimal. I study both static and dynamic security design problems, and show that these two types of problems are equivalent. I use moral hazard in mortgage lending as a recurring example, but my results apply to other corporate finance and principal-agent problems.

1. INTRODUCTION

Debt contracts are widespread, even though debt encourages excessive risk taking. In this paper, I show that debt is the optimal security design in a model in which both reduced effort and excessive risk-taking are possible, even though debt leads to excessive risk taking. In the model, the seller of the security can alter the probability distribution of outcomes in arbitrary ways. This allows the seller to both alter the mean value of the outcome (“effort”) and change the other moments of the distribution of outcomes (“risk-shifting”). To minimize the welfare losses arising from this moral hazard, the security’s payout must be designed to minimize variance. Debt securities are optimal because, among all limited-liability securities with the same expected value, they have the least variance.

The model is motivated by settings in which debt contracts are prevalent and both reduced effort and risk-shifting are possible. For example, in residential mortgage origination, lenders might be able to both underwrite loans more or less diligently (effort) and use private information to

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choose more or less risky borrowers (risk-shifting). Prior to the 2008 financial crisis, mortgage lenders sold debt securities, backed by mortgage loans, to outside investors. The issuance of these securities may have weakened the incentives of mortgage lenders to lend prudently. Despite this effect, I argue that debt can be optimal, because debt securities balance the need to encourage effort with the need to avoid risk-shifting.

Many elements of the model are standard in the security design literature. The security is the portion of the asset value received by the outside investors, and is subject to limited liability constraints. If the seller retains a levered equity claim, she has sold a debt security. There are gains from trade, meaning that the outside investors value the security more than the seller does, holding the distribution of outcomes fixed. Both the outside investors and the seller are risk-neutral.

The key non-standard element of the model is a flexible form of moral hazard, which builds on the work of Holmström and Milgrom [1987]. The seller, through her actions, can create a “zero-cost” distribution of outcomes, which she will do if she has no stake in the outcome. If the seller creates any other probability distribution, she incurs a cost. In my benchmark model, the cost to the seller of choosing a probability distribution \( p \) is proportional to the Kullback-Leibler divergence (or “relative entropy”) of \( p \) from the zero-cost distribution. Under this assumption, the combined effects of reduced effort and risk-shifting can be summarized by one statistic, the variance of the security payoff. The gains from trade are proportional to another statistic, the mean security payoff. Debt securities maximize mean-variance tradeoffs over the set of limited liability securities, and are therefore optimal in this benchmark model.

This mean-variance tradeoff illustrates a key distinction between my model and the existing security design literature. The classic paper of Jensen and Meckling [1976] argues that debt securities are good at providing incentives for effort, but create incentives for risk-shifting, while equity securities avoid risk-shifting problems, but provide weak incentives for effort. A natural conjecture, based on these intuitions, is that when both risk-shifting problems and effort incentives are important, the optimal security will be “in between” debt and equity. In my benchmark model, contrary to this intuition, a debt security is optimal.

The argument of Jensen and Meckling [1976] that debt is best for inducing effort relies on a restriction to monotone security designs. The “live-or-die” result of Innes [1990] (see the appendix, section §A, Figure A.1) shows that when the seller can supply effort to improve the distribution of outcomes (in a monotone likelihood ratio property sense), it is efficient to give the seller all of the asset value when the asset value is high, and nothing otherwise. A revised intuition, which I

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1Throughout the paper, I will use she/her to refer to the seller and he/his to the buyer of the security. No association of the agents to particular genders is intended.
formalize in section §4, is that the securities (including debt) that optimally balance encouraging effort and avoiding risk-shifting are “in between” the live-or-die security and equity.²

I also analyze larger classes of cost functions. When the cost function is not the KL divergence, but instead another $\alpha$-divergence, the optimal security designs exist on a continuum, with the “live-or-die” security at one end, equity at the other, and debt in the middle. In some cases, the security design is upward sloping, and can be thought of as a mix of equity and debt. In other cases, the optimal security design is downward sloping. In these cases, restricting security designs to be monotone for the buyer restores the optimality of debt.

Both the KL divergence and the other $\alpha$-divergences are part of a broader class of divergences, the invariant divergences. For this class of divergences, I show that debt securities, and mixtures of debt and equity, are approximately optimal. The approximation I use applies when the moral hazard and gains from trade are small relative to scale of the assets. It is appropriate in settings in which the divergence, in utility terms, between a well-designed contract and a poorly designed contract is comparable to the seller’s “value added.” I describe the approximation in more detail, and discuss when it is and is not appropriate, in section §5. Under this approximation, debt is first-order optimal, meaning that debt securities are a detail-free way to achieve nearly the same utility as the optimal security design. Mixtures of debt and equity, which correspond to the optimal contracts for $\alpha$-divergences, are second-order optimal for all invariant divergences. This can be interpreted as a “pecking order,” in which the security design grows more complex as the size of both the moral hazard problem and gains from trade grow, relative to the scale of the assets.

Finally, I provide a second micro-foundation for the security design problem with the KL divergence cost function, using a dynamic model. I show that a continuous-time moral hazard problem, similar to Holmström and Milgrom [1987], is equivalent to the static moral hazard problem. The equivalence of the static and dynamic problems provides an intuitive explanation for how the seller can create any probability distribution of outcomes. The key distinction between the dynamic models I discuss and the principal-agent models of Holmström and Milgrom [1987] is limited liability. In Holmström and Milgrom [1987], linear contracts for the seller (agent) are optimal, because they induce the seller to take the same (efficient) action each period. In my model, because of limited liability, the only way to implement the efficient action at every state and time is to offer the seller a very large share of the asset value. However, offering the seller a large share of the asset value limits the gains from trade. It is preferable to pay the seller nothing in the worst states of the world, and then at some point offer a linear payoff. Even though this design does not induce the seller to take the efficient action at every state and time, it achieves more gains from trade. The design for the retained tranche that I have just described, levered equity, corresponds to selling a debt security.

²A similar result, derived from a robust contracting framework, appears in Antic [2014].
The benchmark model in this paper takes the idea of flexibility in moral hazard problems to an extreme, allowing the seller to create any probability distribution of outcomes, subject to a cost. This approach to moral hazard problems was introduced by Holmström and Milgrom [1987]. It is conceptually similar to the notion of flexible information acquisition, emphasized in Yang [2013].

However, in this paper, the cost of choosing a probability distribution should be interpreted as a cost associated with the actions required to cause that distribution to occur (underwriting or not underwriting mortgage loans, for example). In the rational inattention literature, which Yang [2013] builds on, gathering or processing information (as opposed to taking actions) is costly. This distinction is blurred in the rational inattention micro-foundation in the appendix, section §E.

In contrast, much of literature on security design with moral hazard allows the seller to control only one or two parameters of the probability distribution. These papers do not find that debt is optimal. In Mehran et al. [2013], bank managers can both shift risk and pursue private benefits, but do this by choosing amongst three possible investments. In Edmans and Liu [2011], who argue that is efficient for the agent (not the principal) to hold debt claims, also have a binary project choice. Closer to this paper is Biais and Casamatta [1999], in which there are three possible states and two levels of effort and risk-shifting. Biais and Casamatta [1999] interpret the optimal contracts over those three states as mixtures of debt and equity. Hellwig [2009] has a two-parameter model with continuous choices for risk-shifting and effort, and finds that a mix of debt and equity are optimal. In his model, risk-shifting is costless for the agent. Fender and Mitchell [2009] have a model of screening and tranche retention, which is a single-parameter model. This paper differs from this literature by allowing for arbitrary outcome spaces, arbitrary probability distributions, and continuous moral hazard choices, which makes deriving general results difficult (Grossman and Hart [1983]), and by considering flexible models of moral hazard. In the appendix, section §D, I discuss how to extend my results to parametric models, relating the framework I develop to this literature.

Innes [1990] advocates a moral-hazard theory of debt, but debt is optimal only when the seller controls a single parameter, and the security is constrained to be monotone. If the security does not need to be monotone, or if the seller controls both the mean and variance of a log-normal distribution, the optimal contract is not debt. In the corporate finance setting, one argument for monotonicity is that a manager can borrow from a third party, claim higher profits, and then repay the borrowed money from the extra contract payments. In addition to the accounting and legal barriers to this kind of “secret borrowing,” the third party might find it difficult to force repayment. In the context of asset-backed securities, where cash flows are more easily verified, secret borrowing

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3This paper also builds on some of the methods of Yang [2013] (see the appendix, section G.15).

4For brevity, I have omitted the result for log-normal distributions from the paper. It is available upon request.
is even less plausible. Another argument in favor of monotonicity concerns the possibility of the buyer (principal, outside shareholders) sabotaging the project. In the context of securitization, the buyer exerts minimal control over the securitization trust and sabotage is not a significant concern.

There is a large literature that justifies debt for reasons other than moral hazard. Papers invoking adverse selection include Nachman and Noe [1994], DeMarzo and Duffie [1999], Dang et al. [2011], Vanasco [2013], and Yang [2013]. In unreported results, I find that the benchmark model of this paper and of Yang [2013] can be combined to produce debt as the optimal contract, whereas other parametric models of moral hazard, when combined with Yang [2013], would not generally result in debt. Other theories of debt include costly state verification (Townsend [1979], Gale and Hellwig [1985]) and explanations based on control or limiting investment (Aghion and Bolton [1992], Jensen [1986], Hart and Moore [1994]).

I begin in section §2 by explaining the benchmark security design problem, whose structure is used throughout the paper. I then show in section §3 that for a particular cost function, debt is optimal, and explain how this relates to a mean-variance tradeoff. Next, I analyze other cost functions in section §4, describing the optimal contracts and showing a related mean-variance tradeoff applies. I will then introduce an approximation in section §5, and show that for an even larger class of cost functions, the same tradeoffs hold in an approximate sense. In section §6 and section §7, I provide micro-foundations for the non-parametric models, from a continuous time model. In the appendix, section §C, I discuss a calibration for the example of residential mortgage lending. In the appendix, section §D, I discuss parametric models, and apply the results in appendix section §E to a model of rational inattention in mortgage lending.

2. Model Framework

In this section, I introduce the security design framework that I will discuss throughout the paper. The problem is close to Innes [1990] and other papers in the security design literature. There is a risk-neutral agent, called the “seller,” who owns an asset in the first period. In the second period, one of \( N + 1 \) possible states, indexed by \( i \in \Omega = \{0, 1, \ldots, N\} \), occurs.\(^5\) In each of these states, the seller’s asset has an undiscounted value of \( v_i \). I assume that \( v_0 = 0 \), \( v_i \) is non-decreasing in \( i \), and that \( v_N > v_0 \).

The seller discounts second period payoffs to the first period with a discount factor \( \beta_s \). There is a second risk-neutral agent, the “buyer,” who discounts second period payoffs to the first period with a larger discount factor, \( \beta_b > \beta_s \). Because the buyer values second period cash flows more than the seller, there are “gains from trade” if the seller gives the buyer a second period claim in exchange for a first period payment. I will refer to the parameter \( \kappa = \frac{\beta_b - \beta_s}{\beta_s} \) as the gains from trade.

\(^5\)Using a discrete outcome space simplifies the exposition, but is not necessary for the main results.
I assume there is limited liability, so that in each state the seller can credibly promise to pay at most the value of the asset. I also assume that the seller must offer the buyer a security, meaning that the second period payment to the buyer must be weakly positive. In this sense, the seller must offer the buyer an “asset-backed security.” When the asset takes on value $v_i$ in the second period, the security pays $s_i \in [0, v_i]$ to the buyer. Following the conventions of the literature, I will say that the security is a debt security if $s_i = \min(v_i, \bar{v})$ for some $\bar{v} \in (0, v_N)$.

To simplify the exposition, I make particular assumptions about the timing of the events and the bargaining power of the agents. I will assume that, during the first period, the seller first designs the security, and then makes a “take-it-or-leave-it” offer to the buyer at price $K$. If the buyer rejects the offer, the seller retains the entire asset. After the buyer accepts or rejects the offer, the seller takes actions that modify the value of the assets (the moral hazard). The first period ends, uncertainty is resolved, and then in the second period payoffs are determined.

This timing convention, which is standard in principal-agent models, is not appropriate for some applications. For example, in mortgage origination, much of the lender’s moral hazard occurs when the loans are being underwritten, before they are sold to outside investors. In the appendix, section §B, I show that this timing of events is not necessary for the main results. This robustness to the timing of events contrasts with models based on adverse selection by the seller, such as DeMarzo and Duffie [1999], in which the timing of events is crucial. In the same appendix section, I also show that allowing the buyer and seller to Nash-bargain over the price, or over both the price and security design, does not alter the main results.

The moral hazard problem occurs when the seller creates or modifies the asset. During this process, the seller will take a variety of actions, and these actions will alter the probability distribution of second period asset values. Following Holmström and Milgrom [1987], I model the seller as directly choosing a probability distribution, $p$, over the sample space $\Omega$, subject to a cost $\psi(p)$. I will focus models in which any probability distribution $p$ can be chosen, which I will call “non-parametric.” In the appendix, section §D, I discuss models in which $p$ must belong to a parametric family of distributions.\footnote{Because the sample space $\Omega$ is a finite set of outcomes, even in the “non-parametric” case, the choice of $p$ can be expressed as a choice over a finite number of parameters. I am using the terms non-parametric and parametric to denote whether the set $M$ of feasible probability distributions is the entire simplex, or a restricted set.}

I will make several assumptions about the cost function $\psi(p)$. First, I assume that there is a unique probability distribution, $q$, with full support over $\Omega$, that minimizes the cost. Second, because I will not consider participation constraints for the seller, I assume without loss of generality that $\psi(q) = 0$. I also assume that $\psi(p)$ is strictly convex and at least twice differentiable. Below, I will impose additional assumptions on the cost function, but first will describe the moral hazard and security design problems.
The moral hazard occurs because the seller cares only about maximizing the discounted value of her retained tranche. When the value of the asset is $v_i$, the discounted value of the retained tranche is

$$\eta_i = \beta_s (v_i - s_i).$$

Because of the assumption that $v_0 = 0$, and limited liability, it is always the case that $\eta_0 = s_0 = 0$. Let $p^i$ denote the probability that state $i \in \Omega$ occurs, under probability distribution $p$. The moral hazard sub-problem of the seller can be written as

$$\phi(\eta) = \sup_{p \in M} \left\{ \sum_{i>0} \eta_i p^i - \psi(p) \right\},$$

where $M$ is the set of feasible probability distributions and $\phi(\eta)$ is the indirect utility function. In the non-parametric case, when $M$ is the set of all probability distributions on the sample space, the moral hazard problem has a unique optimal $p$ for each $\eta$. Moreover, the smoothness and convexity of $\psi(p)$ guarantee that this optimal policy, $p(\eta)$, is itself differentiable with respect to $\eta$. In contrast, for the parametric case, there may be multiple $p \in M$ that achieve the same optimal utility for the seller. The results I derive for parametric models, in appendix section §D, do not depend on how the seller chooses amongst her optimal actions.

The buyer cannot observe $p$ directly, but can infer the seller’s choice of $p$ from the design of the retained tranche $\eta$. At the security design stage, the buyer’s valuation of a security $s$ is determined by both the structure of the security and the buyer’s inference about which probability distribution the seller will choose, $p(\eta)$. Without loss of generality, I will define the units of the seller’s and buyer’s payoffs so that $\beta_s \sum_i v_i p^i(\beta_s v_i) = 1$. That is, if the seller retains the entire asset, and takes actions in the moral hazard problem accordingly, the discounted asset value is one. I use this convention to ensure that the units correspond to a quantity that is at least potentially observable: the value of the assets, if those assets are retained by the seller. This convention is useful in the calibration of the model in the appendix, section §C.

Let $s_i(\eta)$ be the security design corresponding to retained tranche $\eta$. The security design problem can be written as

$$U(\eta^*) = \max_{\eta} \left\{ \beta_b \sum_{i>0} p^i(\eta)s_i(\eta) + \phi(\eta) \right\},$$

subject to the limited liability constraint that $\eta_i \in [0, \beta_s v_i]$. From the seller’s perspective, when she is designing the security, she internalizes the effect that her subsequent choice of $p$ will have on the
buyer’s valuation, because that valuation determines the price at which she can sell the security. The security serves as a commitment device for the seller, providing an incentive for her to choose a favorable \( p \). This commitment is costly, however, because allocating more of the available asset value to the retained tranche necessarily reduces the payout of the security, reducing the gains from trade.

Many of the results discussed in this paper are derived using perturbation arguments. Any infinitesimal perturbation to the security design (and therefore retained tranche) has two effects on the seller’s utility in the security design problem. The first effect is the “direct” effect, which changes the seller’s utility by transferring more or less expected value from the seller to the buyer. In general, the size of this effect is controlled by the gains from trade parameter, \( \kappa \). The second effect is the “indirect” effect, which changes the buyer’s valuation of the security, through the change in the seller’s behavior in the moral hazard problem. There is no “indirect” effect on the seller’s utility in the moral hazard problem, because the seller is maximizing her utility in the moral hazard problem when she chooses the probability distribution (the envelope theorem). The size of the indirect effect is controlled by the moral hazard parameter, \( \theta \).

Consider a differentiable perturbation around the optimal security design, \( \eta(\epsilon) \), with \( \eta(0) = \eta^* \), that is feasible for some \( \epsilon > 0 \). As mentioned above, in the non-parametric models that are the focus of this paper, \( p(\eta) \) is differentiable. In this case, the two effects of a perturbation can be summarized by the following first-order optimality condition with respect to \( \epsilon \), the size of the perturbation:

\[
\frac{\partial U(\eta(\epsilon))}{\partial \epsilon} \bigg|_{\epsilon=0^+} = -\kappa \sum_{i \in \Omega} p^i(\eta^*) \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon=0^+} + \beta_b \sum_{i,j \in \Omega} s^j \frac{\partial p^j(\eta)}{\partial \eta_i} \bigg|_{\eta=\eta^*} \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon=0^+} \leq 0.
\]

Below, I will further decompose the indirect effect into an indirect effect due to a change in effort and an indirect effect due to a change in risk-shifting. First, however, I will describe the cost functions that I will be studying in more detail.

As discussed earlier, the cost function \( \psi(p) \) is convex and minimized at \( \psi(q) = 0 \). It follows that the cost function can be written as a divergence\(^7\) between \( p \) and the zero-cost distribution, \( q \), defined for all \( p, q \in M \):

\[
\psi(p) = D(p \| q).
\]

There are many divergences that have been defined in the information theory literature (e.g. Ali and Silvey [1966], Csiszár [1967], Amari and Nagaoka [2007]). In section §3, I begin the paper

\(^7\)A “divergence” is similar to a distance, except that there is no requirement that it be symmetric between \( p \) and \( q \), or that it satisfy the triangle inequality.
by focusing on a particular divergence, the Kullback-Leibler divergence. The KL divergence, also
called relative entropy, is defined as

$$D_{KL}(p||q) = \sum_{i \in \Omega} p^i \ln \left( \frac{p^i}{q^i} \right).$$

The KL divergence has the assumed convexity and differentiability properties, and also guarantees
that the $p$ chosen by the seller will be mutually absolutely continuous with respect to $q$. The KL
divergence has been used in a variety of economic models, notably Hansen and Sargent [2008],
who use it to describe the set of models a robust decision maker considers. It also has many
applications in econometrics, statistics, and information theory, and the connection between the
security design problem and these topics will be discussed later in the paper. I will show that when
the cost function is proportional to the KL divergence, debt is the optimal security design.

The KL divergence is a member of the family of $\alpha$-divergences. These divergences are parametrized
by a real number, $\alpha$, which controls how the curvature of the divergence changes as $p$ moves away
from $q$. The $\alpha$-divergences can be written, whenever $|\alpha| \neq 1$, as

$$D_\alpha(p||q) = \sum_{i \in \Omega} \frac{4}{1-\alpha^2} q^i \left( 1 - \left( \frac{p^i}{q^i} \right)^{\frac{1}{2}(1-\alpha)} \right) + \frac{1}{2} (1-\alpha) \left( \frac{p^i}{q^i} - 1 \right).$$

The limits of $\alpha \to -1$ and $\alpha \to 1$ correspond to the KL divergence and the “reversed” KL
divergence, respectively.\footnote{Other authors use different sign conventions or scaling for the $\alpha$ parameter.} For this class of divergences, in section §4 I will show that, for $\alpha \leq -1$, the optimal contracts are mixtures of debt and equity. Commonly discussed $\alpha$-divergences include
the Hellinger distance ($\alpha = 0$) and the $\chi^2$-divergence ($\alpha = -3$).

I will also discuss a more general class of divergences, that contains the $\alpha$-divergences, known
as the “$f$-divergences”. This class of divergences can be written as

(2.4) $$D_f(p||q) = \sum_{i \in \Omega} q^i f \left( \frac{p^i}{q^i} \right),$$

where $f(u)$ is a convex function on $\mathbb{R}^+$, with $f(1) = 0$ and $f(u) \geq 0$. I will limit my discussion
to sufficiently differentiable $f$-functions, for mathematical convenience, and use the normalization
that $f''(1) = 1$. The $f$-divergences are analytically convenient because they are additively separable (or “decomposable”) across states.

The most general class of divergences that I will discuss are the “invariant divergences,” which
contain the $f$-divergences, along with other divergences that are not additively separable, such as
the Chernoff and Bhattacharyya distances. Invariant divergences are defined by their invariance
with respect to sufficient statistics (Chentsov [1982], Amari and Nagaoka [2007]). The exact definition of an invariant divergence is rather technical; for our purposes, what is special about these divergences is that, up to second order, they resemble the KL divergence, and up to third order, they resemble the $\alpha$-divergences. In section §5, I will define this “resemblance” more precisely, and define how a security design can be “approximately optimal.” I will then show that debt, or mixtures of debt and equity, are approximately optimal as a result.

To summarize, the divergences I discuss are related in the following way:

$$KL \in \alpha\text{-divergences} \subset f\text{-divergences} \subset \text{Invariant Divergences} \subset \text{All Divergences}.$$ 

The KL divergence, and the broader class of invariant divergences, are interesting because they are closely related to ideas from information theory. In the appendix, section §E, I illustrate this in a model based on rational inattention (Sims [2003]), in which the cost function is related to the KL divergence. The KL divergence cost function can also be micro-founded from a dynamic moral hazard problem. In section §6, I show that a large class of continuous time problems are equivalent to the static moral hazard problem with a divergence cost function, and show that in a particular case, that divergence is the KL divergence. In section §7, I extend this analysis to a more general class of continuous time problems and show that they are related, in a certain sense, to static moral hazard problems with invariant divergence cost functions.

I will refer throughout the paper to “effort” and “risk-shifting” as separate components of the moral hazard problem. Next, I will define “effort” and “risk-shifting” formally, and clarify the connection between this framework and more conventional models of moral hazard. “Effort” is the change in the discounted expected value of the assets. The “cost” of effort is the solution to the problem

$$c(e) = \inf_{p \in M} D(p||q),$$

subject to the constraint that

$$\beta \sum_{i \in \Omega} (p^i - q^i)v_i = e,$$

where $e$ is the level of effort. In the model of Innes [1990], the seller is restricted to choosing from a family of probability distributions that satisfy a monotone likelihood ratio property. As a result, effort, defined in this way, is one-to-one with the choice variable in Innes [1990].

Given a retained tranche $\eta$, define the effort it induces as $e(\eta)$, and let $p_{\eta}(e)$ denote the probability distribution that minimizes the cost of that effort level. Note that, if the security design (and therefore retained tranche) is an equity claim, then $p_{\eta}(e(\eta)) = p(\eta)$ (this follows from the first-order condition of the moral hazard problem). Viewed a different way, for any retained tranche $\eta$, there is an “equivalent equity share”, $\gamma(\eta)$, for the seller that would induce the same amount of
effort: $e(\eta) = e(\gamma(\eta)\beta s v)$. This equity share is not necessarily feasible— if $\eta$ induces a very high or very low level effort, the equivalent equity share might be more than 100% or less than 0% of the asset value.

In models with more flexible moral hazard, effort is not one-to-one with the choices of the agent. In these models, we can define “risk-shifting” as the actions that the agent takes which change the probability distribution of outcomes without changing the expected value of asset. This includes actions that change the higher moments of the asset distribution, and also actions that keep the distribution of asset values constant, but move probability between states with the same asset value ($i, j \in \Omega$ with $v_i = v_j$).

Using these definitions of effort and risk-shifting, I decompose the utility in the security design problem (equation (2.2)) and the indirect effect of any security design perturbation (equation (2.3)) into effort and risk-shifting components.

\[ U(\eta) = \beta_b \sum_{i \in \Omega} q^i v_i + \left[ \frac{\beta_b e(\eta) - c(e(\eta))}{\beta_s} - \kappa \sum_{i \in \Omega} p^i_e(e(\eta))\eta_i \right] + D(p_e(e(\eta))||q) - D(p(\eta)||q) - \kappa \sum_{i \in \Omega} [p^i(\eta) - p^i_e(e(\eta))]\eta_i. \]

The risk-shifting component is always weakly negative, and is strictly negative if $p_e(e(\eta)) \neq p(\eta)$.

The indirect effect of any security design perturbation (equation (2.3)) can be decomposed into the effect that would occur if the retained tranche were equal to its equivalent equity security, both before and after the perturbation, and the effect due to the differences between the retained tranche and its equivalent equity security. I will interpret these effects as the indirect effects on effort and

Lemma 1. In the security design problem, the utility associated with using a retained tranche $\eta$ can be written as

\[ U(\eta) = \beta_b \sum_{i \in \Omega} q^i v_i + \left[ \frac{\beta_b e(\eta) - c(e(\eta))}{\beta_s} - \kappa \sum_{i \in \Omega} p^i_e(e(\eta))\eta_i \right] + D(p_e(e(\eta))||q) - D(p(\eta)||q) - \kappa \sum_{i \in \Omega} [p^i(\eta) - p^i_e(e(\eta))]\eta_i. \]

The risk-shifting component is always weakly negative, and is strictly negative if $p_e(e(\eta)) \neq p(\eta)$.

The indirect effect of any security design perturbation (equation (2.3)) can be decomposed into the effect that would occur if the retained tranche were equal to its equivalent equity security, both before and after the perturbation, and the effect due to the differences between the retained tranche and its equivalent equity security. I will interpret these effects as the indirect effects on effort and
risk-shifting, respectively:

\[ \beta_b \sum_{j \in \Omega} s_j^* \frac{dp^j(\eta(\epsilon))}{d\epsilon} \bigg|_{\epsilon=0^+} = \frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{dc(\eta(\epsilon))}{d\epsilon} \bigg|_{\epsilon=0^+} - \]

\[ \beta_b \sum_{j \in \Omega} \frac{dp^j(\eta(\epsilon))}{d\epsilon} \bigg|_{\epsilon=0^+} \left( \eta_j^* - \gamma(\eta^*) \beta_s v_j \right) . \]

\[ \text{indirect effect on risk shifting} \]

\[ \text{indirect effect on effort} \]

Proof. See appendix, section G.1.

This decomposition is not unique; there are many other ways of decomposing the utility and the indirect effects into different components. However, this particular decomposition illustrates several key features of the model. First, the seller receives an endowment, which represents the utility the seller can achieve by selling everything to the buyer. Second, the seller responds to her incentives by choosing an effort level. Effort has a benefit, improving the expected value of the assets, but is associated with two costs. The first cost of effort is the cost to the seller of achieving that level of effort, \( c(e) \). The second cost is that creating incentives for effort is costly, because it requires the seller to retain cashflows that could have been sold to the buyer, and therefore forfeits some potential gains from trade. Lastly, risk-shifting has only (ex-ante) costs, and no benefits. Risk-shifting involves the seller taking privately costly actions; the difference between the two divergences in the above decomposition is always weakly negative, by construction. Risk-shifting also reduces the gains from trade, by diverting expected value from the buyer to the seller. It follows that, under an optimal security design, the effort component is weakly positive, and strictly positive if the optimal security design is strictly preferred to selling everything.

This decomposition connects the flexible moral hazard framework used in this paper to other models of moral hazard. Using this definition of effort and risk-shifting, an equity contract causes no utility loss due to risk-shifting, because (as mentioned above) \( p_e(c(\eta)) = p(\eta) \), consistent with the argument of Jensen and Meckling [1976]. However, equity contracts might not be a very efficient way to induce effort by the seller. If the effort level is one-to-one with the seller’s choices (as in Innes [1990]), there is no possibility of risk-shifting (\( p_e(c(\eta)) = p(\eta) \) for all \( \eta \)), and this framework reduces to the classic model of moral hazard.

Moral hazard models with two choice parameters, such as Hellwig [2009], allow the seller to risk-shift in one dimension, while also incorporating an effort choice. The non-parametric model of moral hazard emphasized in this paper extends these models by allowing more dimensions of risk-shifting. In models with only one dimension of risk-shifting, if there are many possible outcomes (i.e. more than the three in Biais and Casamatta [1999]), there will in general be contracts other
than equity contracts that also incur zero risk-shifting costs. In contrast, in the non-parametric model of moral hazard, equity contracts are the only contracts that incur zero risk-shifting costs.

The decomposition of the indirect effect of a perturbation into an effect that would occur for the equivalent equity securities (“effort”) and an effect that occurs because of the differences between the retained tranche and equivalent equity security (“risk-shifting”) illustrates the externalities associated with the seller’s choices in the moral hazard problem. The buyer benefits from an increase in the seller’s effort, assuming that the seller’s equivalent equity share is less than one hundred percent. At the same time, the buyer can benefit or be harmed by the change in the seller’s risk shifting behavior, depending on whether the change in the security design induces more or less risk shifting. I will show in the following sections that the effect of a perturbation to the security design on risk shifting depends on whether the security becomes more or less equity-like.

The models described in the paper use divergences to create cost functions, which rules out two interesting cases: free disposal of output by the seller, and free risk-shifting. Free disposal of output by the seller is a common assumption in security design problems, and is used to justify restricting the set of securities to designs for which the seller’s payoff is weakly increasing in the asset value. Free disposal of output does not change any of the results in the paper— all of the optimal security designs without free disposal have monotone payoffs for the seller, and are therefore still optimal among the set of monotone security designs. I discuss this more in the appendix, section §F.

Free risk-shifting is the assumption that only effort, and not risk-shifting, is costly for the agent. Formally, this would require that \( D(p||q) = D(p'||q) \) for all \( p, p' \) with the same expected value. Technically, the assumptions of strict convexity for \( D(p||q) \) and that \( D(p||q) = 0 \) only if \( p = q \) both rule out this case. However, using lemma 1, the analysis in this case is straightforward. As risk-shifting becomes free, concerns about risk-shifting dominate concerns about effort, and equity contracts are optimal. This result is closely related to Ravid and Spiegel [1997] and to Carroll [2015], and is also shown in the appendix, section §F.

In this section, I have introduced the framework that I will use throughout the paper. In the next section, I analyze the benchmark model, in which the cost function is the KL divergence.

3. The Benchmark Model

In this section, I discuss the non-parametric version of the model, in which the set \( M \) of feasible probability distributions is the set of all probability distributions on \( \Omega \). I assume that the cost function is proportional to the KL divergence between \( p \) and \( q \),

\[
\psi(p) = \theta D_{KL}(p||q).
\]
where $\theta$ is a positive constant that controls how costly it is for the seller to have $p$ deviate from $q$. Larger values of $\theta$ make deviations more costly, and therefore reduce the size of the moral hazard problem.

I will show that the optimal security design is a debt contract. In the text, I will outline the proof, using a perturbation argument; a complete proof can be found in the appendix, section G.5.\footnote{This perturbation argument builds on the suggestions of an anonymous referee.}

I will start by discussing the first-order condition of the moral hazard problem. The KL divergence cost function becomes infinitely sloped at the boundaries of the simplex, and therefore guarantees an interior solution to the moral hazard problem, equation (2.1), for all $\eta$. The KL divergence is also convex, consistent with the assumptions described in the previous section. As a result, the first-order condition in the moral hazard problem must hold. For any $i > 0$, we have

$$\eta_i = \theta \left( \ln \left( \frac{p_i}{q_i} \right) - \ln \left( \frac{p_0}{q_0} \right) \right).$$

Intuitively, if the seller receives a high payoff in state $i$, she will increase the probability of state $i$ relative to state 0, in which she receives zero payoff.

From this first-order condition, we can observe that the semi-elasticities of the relative probabilities $p^i(\eta)$ and $p^0(\eta)$ to the payoff $\eta_i$ satisfy

$$\frac{\partial \ln(p^i(\eta))}{\partial \eta_i} - \frac{\partial \ln(p^0(\eta))}{\partial \eta_i} = \theta^{-1}. \quad (3.1)$$

This constant difference of semi-elasticities property is part of what is special about the KL divergence. It is constant in two respects; first, the difference of the elasticities does not depend on how far $p(\eta)$ is from $q$, and second, it is symmetric across the states $i \in \Omega$. The $\alpha$-divergences that will be discussed in the next section relax the first of these properties—the elasticity will depend on how far the endogenous probability distribution is from the zero-cost distribution. The entire class of invariant divergences, which are used throughout the paper, share the second property, imposing a sort of symmetry across states of the world (this is essentially the meaning of “invariant”).

Using this property, we can construct perturbations of the retained tranche (and therefore the security design) that changes the probability in two different states, $p^i$ and $p^j$, with $i > 0$ and $j > 0$, while leaving all other probabilities unchanged. Let $\eta^*$ be the optimal design for the retained tranche. Suppose that, starting from $\eta^*$, we increase $\eta_i$ by an amount $\frac{\epsilon \cdot \eta^*}{p^i(\eta^*)}$, while decreasing $\eta_j$ by an amount $\frac{\epsilon \cdot \eta^*}{p^j(\eta^*)}$. Conjecture that this perturbation, for infinitesimal values of $\epsilon$, increases $p^i$ and decreases $p^j$ by $\theta^{-1}\epsilon$, while leaving all other probabilities, and in particular $p^0$, unchanged. We can verify this conjecture by observing that equation (3.1) above is satisfied for all states, and that the sum of the probabilities across states remains equal to one.
Having constructed this perturbation, I now turn to the security design problem. Consider the following property of debt: for a security $s$ to be a debt, there must be no pairs $s_i$ and $s_j$, with $i \neq j$, such that $s_j < v_j$ and $s_j < s_i$. This property requires that if the limited liability constraint does not bind in either state $i$ or state $j$, the security values must be equal, and if the constraint binds only in one of the two states, the payoff in that state must be smaller than in the “flat” part of the debt contract. It is essentially the definition of a debt contract, subject to the caveat that “selling everything” and “selling nothing” also have this property.

Suppose that the optimal security design $s^*$ does not have this property (and therefore is not debt). For this to be true, there must be no perturbation of the security design that is feasible and can improve the seller’s utility in the security design problem. Using the perturbation described above, I will show that such a perturbation does exist, and therefore that the optimal contract is a debt (or selling everything/nothing, which are ruled out in the proof in the appendix).

We have supposed that, for the optimal security design $s^*$, there is a pair of states $i, j \in \Omega$, $i \neq j$, with $s^*_j < v_j$ and $s^*_j < s^*_i$. Now imagine that we increase $s_j$ by $\beta s^{-1} \epsilon p(v^*_j)$ while decreasing $s_i$ by $\beta s^{-1} \epsilon p(v^*_i)$. The values of the retained tranche in those states, $\eta_i$ and $\eta_j$, move opposite the security design and are perturbed in exactly the manner discussed above. Note that, because $s^*_j < v_j$ and $s^*_i > s^*_j$, this perturbation does not violate the limited liability constraints.

The effect of this perturbation on the utility in the security design problem is described by equation (2.3) in the previous section. We can see that there is no “direct effect” of this perturbation; holding the probability distribution the seller chooses fixed, the perturbation does not affect the expected value of the security design. The perturbation does increase the probability of state $i$ by $\theta^{-1} \epsilon$, and it decreases the probability of state $j$ by $\theta^{-1} \epsilon$, leaving the probability of all other states the same. Therefore, the “indirect” effect is $\theta^{-1} (s^*_i - s^*_j)$, which was assumed to be greater than zero. It follows that this perturbation improves the seller’s utility, and therefore the optimal contract must be a debt, selling everything, or selling nothing.

This argument can be thought of as showing that the security design should be “flat wherever possible.” If the security pays the buyer more in state $i$ than in state $j$, the seller will inefficiently act to ensure that state $i$ is less likely than state $j$. Flat securities eliminate this problem, but because of the limited liability constraints, the security cannot be completely flat, unless it pays nothing at all and foregoes all of the gains from trade. Debt securities are the optimal compromise: they have positive expected value, capturing some gains from trade, but are flat wherever possible, minimizing inefficient actions by the seller.

After introducing the formal result, I discuss an alternative (but equivalent) perspective on why the KL divergence leads to contracts that are flat wherever possible. I will also return to the decomposition between effort and risk-shifting introduced in the previous section. The following
proposition summarizes this perturbation argument, rules out selling everything and selling nothing, and also establishes a result about the face value of the debt contract.

**Proposition 1.** In the non-parametric model, with the cost function proportional to the Kullback-Leibler divergence, the optimal security design is a debt contract,

\[ s_j^* = \min(v_j, \bar{v}), \]

for some \( \bar{v} > 0 \). The face value of the debt satisfies

\[ \beta_b \bar{v} - \beta_b \sum_{i \in \Omega} p_i^j(q^*) s_i^* = \kappa \theta. \]

If the highest possible asset value is sufficiently large (\( v_N > \sum_i q_i v_i + \frac{\kappa}{\beta_b} \theta \)), then \( \bar{v} < v_N \).

**Proof.** The results are proven in the proof of proposition 3. \( \square \)

The result in proposition 1 shows that debt is optimal, for any full-support zero-cost distribution \( q \). The condition that \( v_N \) be “high enough” is weak. If it was not satisfied for some sample space \( \Omega \) and zero-cost distribution \( q \), one could include a new highest value \( v_{N+1} \) in \( \Omega \), occurring with vanishingly small probability under \( q \), such that the condition was satisfied. Intuitively, the sample space must contain high enough values to observe the “flat” part of the debt security.

The perturbation argument described above lead to the conclusion that the security design should be flat wherever possible. A different way to view the same idea, which is mathematically equivalent, can be derived by analyzing the indirect effect described above. When the seller is considering a perturbation to the security design, she must take into account the effect that this perturbation has on the buyer’s beliefs about the seller’s actions in the moral hazard problem. In the security design first-order condition, equation (2.3), the term \( \frac{\partial p(k)}{\partial \eta} \bigg|_{\eta=\eta^*} \) captures this effect. Differentiating the first-order condition of the moral hazard problem, we can observe that, for \( i, j \in \Omega \), with \( i > 0 \) and \( j > 0 \),

\[ 1(i = j) = \sum_{k>0} \frac{\partial \psi(p)}{\partial p^i \partial p^k} \bigg|_{p=p^*} \frac{\partial p^k(\eta)}{\partial \eta_j} \bigg|_{\eta=\eta^*}. \]

In matrix terms, the effect of a security design change on the probabilities is the inverse of the matrix of second derivatives of the cost function. An analogous result is common in principal-agent problems; the agent’s second-order condition is often related to the principal’s first-order condition.

With the KL divergence, the first derivative of the cost function is, for any \( i > 0 \),

\[ \frac{\partial \psi(p)}{\partial p^i} = \theta (\ln(p^i) - \ln(p^0)). \]
Differentiating this again, we have
\[ \frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} = \frac{\theta}{p^i} \mathbf{1}(i = j) + \frac{\theta}{p^j} = \theta g_{ij}(p), \]
where \( g_{ij}(p) \) is the Fisher information matrix.

The Fisher information matrix appears in numerous applications related to econometrics and statistics. In particular, the inverse Fisher information matrix, which I will denote \( g^{-1}(p) \), appears in the Cramér-Rao bound. For non-parametric models, this bound is an equality. That is, for any security design \( s \) and probability distribution \( p \), the variance of the security payout, \( V^p[s] \), is equal to
\[ V^p[s] = \sum_{i,j \in \Omega} s_i s_j g_{ij}(p). \]
In the non-parametric model, the Cramér-Rao equality is easy to derive— one can invert the definition of the Fisher information matrix given above and verify the result.

Armed with this fact, and our just-derived result that the matrix of second derivatives (the “Hessian”) of the cost function is proportional to the Fisher information matrix, we can re-interpret the security design first-order condition. We can rewrite equation (2.3) as
\[ \kappa \frac{\partial}{\partial \epsilon} E^p(\eta^*)[\beta_s s(\epsilon)]|_{\epsilon = 0^+} - \frac{1}{2} \frac{\partial}{\partial \epsilon} V^p(\eta^*)[\beta_s s(\epsilon)]|_{\epsilon = 0^+} \leq 0, \]
where \( E^p(\eta^*) \) denotes the expectation, and \( s(0) \) is the optimal security design, \( s^* \). Note that the security design, \( s(\epsilon) \), is affected by the perturbation, but the probability distribution, \( p(\eta^*) \), is not.

This equation offers a different perspective on why the KL divergence cost function leads to debt contracts as the optimal security design. The perturbation argument discussed earlier lead to the conclusion that the optimal contract should be flat wherever possible. This equation suggests a reason for this— flat contracts minimize variance.\(^{10}\) Intuitively, if the seller perturbs the security design in a way that (under a fixed probability distribution) holds the expected value constant, but increases the variance of the payout, she has given herself opportunities to take advantage of the buyer in the moral hazard problem. The buyer observes this, and is willing to pay less for the security. The seller internalizes this effect when designing the security, in effect using the flatness of the security as a commitment to avoid acting inefficiently in the moral hazard problem.

Examining the equity, live-or-die, and debt securities shown in the appendix, Figure A.1, it is clear why the debt security minimizes the variance of the payout, among all limited-liability securities with the same expected value— because they are as flat as possible. The proof of proposition 1

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\(^{10}\)Shavell [1979] mentions that flat contracts minimize variance, in a context without limited liability. A related result with limited liability can be found in Plantin [2014].
above can be understood as showing both that the variance-minimizing security is a debt contract, and that debt is optimal in the security design problem.

We can take this argument one step further and decompose the “indirect effect” into effort-only and risk-shifting components.

**Corollary 1.** With the KL divergence cost function, the indirect, effort-only effect is

$$\frac{\beta_b}{\beta_s}(1 - \gamma(\eta^*)) \frac{de(\eta(\epsilon))}{d\epsilon}|_{\epsilon=0^+} = \theta^{-1} \frac{\beta_b}{\beta_s}(1 - \gamma(\eta^*)) \frac{\partial}{\partial \epsilon} \text{Cov}^{p(\eta^*)}[\eta(\epsilon), \beta_s v]|_{\epsilon=0^+},$$

where \(\text{Cov}^{p(\eta^*)}\) denotes covariance. The indirect effect on risk shifting is

$$-\frac{\beta_b}{\beta_s} \sum_{j \in \Omega} dp_j^j(\eta(\epsilon))|_{\epsilon=0^+} (\eta_j^* - \gamma(\eta^*) \beta_s v_j) = -\frac{1}{2} \theta^{-1} \frac{\beta_b}{\beta_s} \frac{\partial}{\partial \epsilon} V^{p(\eta^*)}[\eta(\epsilon) - \gamma(\eta^*) \beta_s v]|_{\epsilon=0^+}.$$

**Proof.** The results are proven in the proof of corollary 3. □

Intuitively, if we perturb the security design to make the seller’s retained tranche more aligned with the value of the underlying assets, this induces the seller to exert more effort. This extra effort benefits the buyer, assuming that the seller is not the full residual claimant. The special property of the KL divergence is that the correct notion of “alignment” is covariance. Similarly, if we perturb the security design to cause the seller’s retained tranche to vary more, relative to the equity tranche that induces the same amount of effort, we create more opportunities for risk-shifting, reducing the value of the buyer’s security. Again, the special property of the KL divergence is that the variance is the correct way to measure this effect.

Several of the assumptions in the benchmark model can be relaxed without altering the debt security result of proposition 1. The lowest possible value, \(v_0\), can be greater than zero. As discussed in the appendix, section §B, the timing of the events and the bargaining power of the agents can be altered without changing the result that debt is optimal. The buyer can be risk-averse, with any increasing, differentiable utility function (this follows from the variance-minimizing property of debt). Finally, although the model is set up as a security design problem, similar results could be obtained in principal-agent and investor-entrepreneur contexts.

The optimal security described in proposition 1 has an interesting comparative static. Define the “put option value” of a debt contract as the discounted difference between its maximum payoff \(\bar{v}\) and its expected value. Proposition 1 states that

(3.2) \(P.O.V. = \beta_b \bar{v} - \beta_b E^{p(\eta^*)}[s^*] = \kappa \theta\).

When the constant \(\theta\) is large, meaning that it is costly for the seller to change the distribution, the put option will have a high value. Similarly, when the gains from trade, \(\kappa\), are high, the put option
will have a high value. For all distributions \( q \), a higher put option value translates into a higher “strike” of the option, \( \bar{v} \), although the exact mapping depends on the distribution \( q \) and the sample space \( \Omega \). Restated, when the agents know that the moral hazard is small, or that the gains from trade are large, they will use a large amount of debt, resulting in a riskier debt security.\(^{11}\)

In this section, I have shown that using the KL divergence cost function leads to debt securities as the optimal contracts. In the next section, I consider alternative cost functions, applying both the perturbation argument and mean-variance intuition developed in this section.

4. The Non-Parametric Model with Invariant Divergences

In this section, I analyze more general classes of divergences as cost functions. First, I will show that among the \( f \)-divergences, the Kullback-Leibler divergence is the only divergence that always results in debt as the optimal security design, allowing for non-monotone security designs, but there are many \( f \)-divergences for which the optimal monotone security design is always a debt security. Second, in the particular case of the \( \alpha \)-divergences, which are a subset of the \( f \)-divergences, I show that the optimal contract is, for some parameter values, a mix of debt and equity.

I begin by assuming that the cost function is proportional to an \( f \)-divergence (defined earlier, see equation (2.4)):

\[
\psi(p) = \theta D_f(p||q),
\]

with an associated \( f \) function that is continuous on \([0, \infty)\) and twice-differentiable on \((0, \infty)\). These divergences are analytically tractable because of the additive separability. That is, the cost of choosing some \( p^i \) is not affected by value of \( p^j, j \neq i \), except through the constraint that probability distributions must add up to one. In some cases, such as the Hellinger distance or KL divergence, the seller’s choice of \( p \) is guaranteed to be interior, but this is not true for all \( f \)-divergences.

Among this family of divergences, the KL divergence is special.

**Proposition 2.** In the non-parametric model, with an \( f \)-divergence cost function, if the optimal security design is debt for all sample spaces \( \Omega \) and zero-cost probability distributions \( q \), then that \( f \)-divergence is the Kullback-Leibler divergence.

**Proof.** See appendix, section G.2. \( \square \)

The statement of proposition 2 shows that the KL divergence is special, in the sense that it is the only continuous and twice-differentiable \( f \)-divergence that always results in debt as the

\(^{11}\)The model also has comparative statics for the zero-effort distribution \( q \), which I will not discuss in detail. Because the endogenous distribution \( p^* \) is not the same as \( q \), perturbations to \( q \) can have ambiguous effects. A mean-preserving spread perturbation to \( q \) can decrease the optimal debt level, because higher volatility increases the value of the put option, or increase it, because it can increase the mean of \( p^* \), decreasing the value of the put option.
optimal security design. The proof uses a perturbation argument, similar to the one employed in the previous section, which I outline below.

Suppose that the solution to the moral hazard problem is interior. The first-order condition in the moral hazard problem, for an arbitrary $f$-divergence and some $i > 0$, is

$$\eta_i = \theta(f'(p^i(\eta) q^i) - f'(p^0(\eta) q^0)).$$

The analog of the difference of elasticities equation used in the previous section (equation (3.1)) is

$$f''(p^i(\eta) q^i) \frac{\partial \ln(p^i(\eta))}{\partial \eta_i} - f''(p^0(\eta) q^0) \frac{\partial \ln(p^0(\eta))}{\partial \eta_i} = \theta^{-1}.$$

For the KL divergence, with $f(u) = u \ln u - u + 1$, we have $uf''(u) = 1$, and this equation reduces to the one introduced previously. For any other $f$-divergence, these terms are not constant.

There is still a perturbation that changes the probabilities in states $p^i$ and $p^j$, while leaving all other probabilities unchanged. Suppose that we increase $\eta_i$ by $\frac{\epsilon}{q^i} f''(p^i(\eta^*) q^i)$, and decrease $\eta_j$ by $\frac{\epsilon}{q^j} f''(p^j(\eta^*) q^j)$. Using the same logic described in the previous section, one can show that this perturbation increases $p^i$ by $\theta^{-1} \epsilon$ and decreases $p^j$ by the same amount, while leaving all other probabilities unchanged.

Now suppose that a debt contract is the optimal security design, for an arbitrary $f$-divergence, and that there are two states associated with the flat part of the debt contract, $i$ and $j$. Consider, as before, a perturbation that decreases the value of the security in state $i$, while increasing the value of the security in state $j$, so that the values of the retained tranche, $\eta_i$ and $\eta_j$, change as described in the previous paragraph. Note that, because we have assumed that the states $i$ and $j$ are associated with the flat part of the debt contract, limited liability does not bind, and this perturbation is feasible.

By construction, the “indirect effect” (see equation (2.3)) of this perturbation is zero. The probability of state $i$ increases by $\theta^{-1} \epsilon$, while the probability of state $j$ decreases by the same amount, and we have assumed that $s_i = s_j$. However, the “direct effect” is not necessary zero. We have

$$\frac{\partial U(\eta(\epsilon))}{\partial \epsilon} = \kappa \left( \frac{p^i(\eta^*)}{q^i} f''(p^i(\eta^*) q^i) - \frac{p^j(\eta^*)}{q^j} f''(p^j(\eta^*) q^j) \right).$$

Of course, if $uf''(u)$ is constant, then this effect is also zero (the KL divergence case). However, in general this will not be the case, and either this perturbation or the “reverse” perturbation (with respect to the states $i$ and $j$) can improve the seller’s utility. The proof of proposition 2 finishes the argument by constructing samples spaces $\Omega$ and zero-cost distributions $q$ such that, for debt to always be optimal, $uf''(u)$ must be constant for all $u \in [0, \infty)$. 
This result depends crucially on the possibility of non-monotone security designs. I have argued in the introduction that, in the context of securitization, there is no particular reason to think that security designs must be monotone. However, in other contexts, following many papers in the security design literature, it may be appropriate to require that security designs result in payoffs that are weakly increasing for both the buyer and the seller. If we impose this assumption, the perturbation logic described above leads to a very different conclusion— that debt securities are optimal as long as $uf''(u)$ is weakly decreasing in $u$.

I will say that a security design is weakly monotone for the buyer if $v_j \geq v_i$ implies that $s_j \geq s_i$. Suppose that $v_j \geq v_i$ and $s_j = s_i$. In this case, $\eta_j \geq \eta_i$, and therefore, by the seller’s first-order condition and the convexity of the $f$ function, $\frac{v^j(\eta)}{q^j} \geq \frac{v^i(\eta)}{q^i}$. That is, because the seller’s payoff is higher in state $j$ than in state $i$, she acts to increase the likelihood of state $j$ relative to state $i$. If $uf''(u)$ is weakly decreasing in $u$, the perturbation analyzed in equation (4.1) (increasing $s_j$ and decreasing $s_i$), starting from a debt security design, reduces the seller’s welfare. Because of the requirement that security designs be monotone, the reverse perturbation (decreasing $s_j$ and increasing $s_i$) is not feasible. As a result, there is no feasible perturbation that can increase welfare, and debt is optimal. The corollary below summarizes the result:

**Corollary 2.** In the non-parametric model, with an $f$-divergence cost function such that $uf''(u)$ is weakly decreasing in $u$, if security designs is required to be monotone for the buyer, then the optimal security design is debt, selling nothing, or selling everything, for all sample spaces $\Omega$ and zero-cost probability distributions $q$.

**Proof.** See appendix, section G.3.

The result of proposition 2 raises another question: absent monotonicity constraints, what are the optimal security designs with this class of cost functions? The logic of the perturbation argument above leads us to conclude that the function $uf''(u)$ plays a critical role in determining the shape of the contract. For a particular sub-class of $f$-divergences, the $\alpha$-divergences, the resulting optimal contracts are easy to characterize. Recall that, for the $\alpha$-divergences,

$$f(u) = \frac{4}{1-\alpha^2}(1 - u^{\frac{1}{2}(1-\alpha)} + \frac{1}{2}(1-\alpha)(u-1)).$$

For these divergences, when $\alpha < -1$, it is possible that the seller will set $p^i = 0$ for some $i \in \Omega$. The proof of proposition 3 deals with this possibility; in the main text, I will assume that $p(\eta)$ is interior in the neighborhood of the optimal security design.

It follows from the iso-elastic nature of these $f$-functions that

$$uf''(u) = 1 - \frac{1 + \alpha}{2} f'(u).$$
The first-order condition of the moral hazard problem implies that, for any retained tranche \( \eta_i \),

\[
\frac{p^j(\eta)}{q^j} f''\left(\frac{p^j(\eta)}{q^j}\right) - \frac{p^i(\eta)}{q^i} f''\left(\frac{p^i(\eta)}{q^i}\right) = \frac{1 + \alpha}{2} \theta^{-1}(\eta_i - \eta_j).
\]

Consider the same perturbation discussed above: increasing \( \eta_i \) by \( \epsilon q^j f''(\frac{p^j(\eta^*)}{q^j}) \), and decreasing \( \eta_j \) by \( \epsilon q^i f''(\frac{p^i(\eta^*)}{q^i}) \). Suppose that this perturbation is feasible. As discussed above, this will increase \( p^i \) by \( \theta^{-1} \epsilon \) and decrease \( p^j \) by the same amount. If the security is not flat, the “indirect effect” is non-zero; we have

\[
\beta_b \sum_{i,j \in \Omega} s^*_j \frac{\partial p^j(\eta)}{\partial \eta_i} \bigg|_{\eta=\eta^*} \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon=0^+} = \theta^{-1} \beta_b (s_i - s_j)
\]

Similarly, as argued above, the “direct effect” is non-zero:

\[
-\kappa \sum_{i \in \Omega} p^j(\eta^*) \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon=0^+} = \kappa \left( \frac{p^j(\eta^*)}{q^j} f''\left(\frac{p^j(\eta^*)}{q^j}\right) - \frac{p^i(\eta^*)}{q^i} f''\left(\frac{p^i(\eta^*)}{q^i}\right) \right)
\]

It follows that if

\[
\frac{\beta_b (s_i - s_j)}{\eta_i - \eta_j} = -\frac{\kappa}{1 + \kappa} \frac{1 + \alpha}{2},
\]

the indirect and direct effects will cancel, and this perturbation will not change the seller’s utility in the security design problem. Therefore, for the optimal security design, for all pairs \( i, j \in \Omega \) such that the limited liability constraints do not bind, the relative slopes of the security design and retained tranche are the same.

For the \( \alpha \)-divergences, the optimal contracts will be straight lines wherever the limited liability constraints do not bind. When \( \alpha = -1 \) (the KL divergence case), we recover the result that the optimal contract is flat when the constraints don’t bind. For \( \alpha < -1 \), the required constant is positive, which implies that both the security design and the retained tranche are upward sloping (in the region where the limited liability constraints do not bind). When \( \alpha > -1 \), the required constant is negative, implying a downward sloping (and therefore non-monotone) security design. These are the \( \alpha \)-divergences for which \( u f''(u) \) is decreasing in \( u \). If the security design was required to be monotone, corollary 2 would apply, and debt (or selling everything/nothing) would be optimal for all \( \alpha \geq -1 \).

The proposition below builds on these ideas, describing the optimal contract for all \( \alpha \).

**Proposition 3.** Define \( s_{\alpha,i} \) as the optimal security design for the problem with an \( \alpha \)-divergence cost function. If \( \alpha < 1 + \frac{2}{\kappa} \), there exists a constant \( \bar{\nu} \geq 0 \) such that
\[ s_{\alpha,i} = \begin{cases} v_i & \text{if } v_i < \bar{v} \\ \max \left[ -\frac{\kappa(1+\alpha)}{2 + \kappa(1-\alpha)} (v_i - \bar{v}) + \bar{v}, 0 \right] & \text{if } v_i \geq \bar{v}. \end{cases} \]

If \( \alpha \geq 1 + \frac{2}{\kappa} \), the optimal security design is the “live-or-die” contract,

\[ s_{\alpha,i} = \begin{cases} v_i & \text{if } v_i < \bar{v} \\ 0 & \text{if } v_i > \bar{v}. \end{cases} \]

When \( \alpha < -3 \), \( \bar{v} \) is strictly greater than zero. In all of these cases, if the highest possible asset value is sufficiently large (\( v_N > \sum q_i v_i + \frac{\kappa}{\beta_b} \theta \)), then \( \bar{v} < v_N \).

**Proof.** See appendix, section G.10. \( \square \)

The optimal security design can be thought of as a mixture of debt and equity (at least when \( \alpha \leq -1 \)), whose slope is determined by the gains from trade parameter \( \kappa \) and the parameter \( \alpha \). For the KL divergence, \( \alpha = -1 \), the contract is always a debt contract, regardless of \( \kappa \). For any \( \alpha > -1 \), the optimal contract is non-monotonic, first increasing up to \( \bar{v} \), then decreasing, and finally paying the buyer zero when the asset value is very high. In Figure A.2, in the appendix, I illustrate the different optimal security designs associated with varying values of \( \alpha \), holding \( \bar{v} \) fixed.

The mean-variance perspective discussed in the previous section can provide a different perspective on why the parameter \( \alpha \) affects the optimal security design in the way it does. As in the previous section, consider the matrix of second derivatives of the \( \alpha \)-divergence cost function, assuming that the solution to the seller’s moral hazard problem is interior. We have

\[ \frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} = \hat{\theta}(p) g_{ij}(\hat{p}(p)), \]

where

\[ \hat{p}^i(p) = \frac{(p^i)^{\frac{1}{2}(\alpha+3)} (q^i)^{-\frac{1}{2}(\alpha+1)}}{\sum_{j \in \Omega} (p^j)^{\frac{1}{2}(\alpha+3)} (q^j)^{-\frac{1}{2}(\alpha+1)}} \]

and

\[ \hat{\theta}(p) = \frac{\theta}{\sum_{j \in \Omega} (p^j)^{\frac{1}{2}(\alpha+3)} (q^j)^{-\frac{1}{2}(\alpha+1)}}. \]

In the KL divergence case (\( \alpha = -1 \)), \( \hat{p} = p \) and \( \hat{\theta} = \theta \). When \( \alpha > -1 \), \( \hat{p}^i \) is monotonically increasing in \( \frac{q^i}{q} \). That is, when \( \alpha > -1 \), \( \hat{p} \) places large weight on states where the seller acts to increase the probability distribution, and small weight on states where the seller acts to decrease the probability distribution. When \( \alpha < -1 \), the reverse is true.
Using the same ideas introduced in the previous section, and these definitions, it follows (see the proof of corollary 3) that the security design first-order condition can be written as

\[
\kappa \frac{\partial}{\partial \epsilon} E^{p(\eta^*)}[\beta_s s(\epsilon)]|_{\epsilon=0^+} - \frac{1}{2} \beta_s \left( \hat{\beta}(p(\eta^*)) \right)^{-1} \frac{\partial}{\partial \epsilon} V^{\hat{p}(p(\eta^*))}[\beta_s s(\epsilon)]|_{\epsilon=0^+} \leq 0.
\]

In the KL divergence case, the expectation in the direct effect and the variance in the indirect effect are taken under the same probability distribution, \( p(\eta^*) \). For other \( \alpha \)-divergences, the variance is taken under \( \hat{p} \neq p(\eta^*) \). Suppose, as proposition 3 shows, that the optimal retained tranche is weakly monotonically increasing in the asset value. In this case, if \( v_i \geq v_j \), then \( \frac{p^i(\eta^*)}{q^i} \geq \frac{p^j(\eta^*)}{q^j} \); that is, the seller acts to increase the probability of good outcomes relative to the probability of bad outcomes. For \( \alpha > -1 \), \( \hat{p}(p(\eta^*)) \) will place even more probability mass on good outcomes, relative to bad outcomes, than \( p(\eta^*) \). For these cost functions, variance is most harmful in the best states of the world. Conversely, for \( \alpha < -1 \), \( \hat{p}(p(\eta^*)) \) will place less mass on good outcomes than \( p(\eta^*) \), implying that variance is more costly in the worst states of the world.

This effect influences the optimal security design, as proposition 3 shows. When \( \alpha > -1 \), and therefore \( \hat{p}(p(\eta^*)) \) places larger weight on the best states of the world than \( p(\eta^*) \), the indirect effect is larger relative to the direct effect, when compared with \( \alpha = -1 \) (the KL divergence case). Put another way, the moral hazard concerns are larger relative to the gains from trade. As a result, the optimal security design gives less to the buyer than a debt contract in the best states of the world. When \( \alpha < -1 \), the reverse is true– in the best states of the world, \( p(\eta^*) > \hat{p}(p(\eta^*)) \), and the direct effect is larger relative to the indirect effect, when compared with \( \alpha = -1 \). In this case, the gains from trade are larger relative to the moral hazard concerns, and the optimal security design gives more cashflows to the buyer than a debt contract. That is, the parameter \( \alpha \) influences the balance of concern about gains from trade and moral hazard across the various states.

This effect occurs because the parameter \( \alpha \) controls the way the curvature of the cost function changes as the seller moves \( p(\eta) \) away from \( q \). Recall that, for all \( f \)-divergences, including the \( \alpha \)-divergences, we normalized the \( f \) function so that \( f''(1) = 1 \). For the \( \alpha \)-divergences, we have

\[
f''(1) = -\frac{1}{2}(\alpha + 3).
\]

When \( \alpha \) is large, the cost function becomes less curved as \( p^i \) becomes large relative to \( q^i \), and more curved as \( p^i \) becomes small relative to \( q^i \). In the best states of the world, the seller increases \( p^i \) relative to \( q^i \) under the optimal contract. Combining these two facts, we can see that if a perturbation increases the variance of the security design in the best states of the world, the seller would easily be able to alter her actions in response. When \( \alpha \) is small, the opposite occurs– the increasing
curvature of the cost function in the best states of the world prevents the seller from responding to perturbations that affect those states.

We can explore this idea further by decomposing the indirect effect into effort and risk-shifting components. The decomposition is essentially identical to the one described in corollary 1, except that the covariances and variances are taken with respect to the probability distribution $\hat{p}(p(\eta^*))$.

**Corollary 3.** With an $\alpha$-divergence cost function, if the solution to the seller’s moral hazard problem is interior, the indirect, effort-only effect is

$$\frac{\beta_b}{\beta_s}(1 - \gamma(\eta^*))\frac{de(\eta(\epsilon))}{d\epsilon}|_{\epsilon=0^+} = \hat{\theta}(p^*(\eta))^{-1}\frac{\beta_b}{\beta_s}(1 - \gamma(\eta^*)) \frac{\partial}{\partial \epsilon} COV_{\hat{p}(p(\eta^*))}(\eta(\epsilon), \beta_s \nu)|_{\epsilon=0^+},$$

and the indirect effect on risk shifting is

$$-\frac{\beta_b}{\beta_s} \sum_{j \in \Omega} dp_j(\eta(\epsilon)) \frac{de(\eta(\epsilon))}{d\epsilon}|_{\epsilon=0^+} = -\frac{1}{2} \hat{\theta}(p^*(\eta))^{-1}\frac{\beta_b}{\beta_s} \frac{\partial}{\partial \epsilon} V_{\hat{p}(p(\eta^*))}[\eta(\epsilon) - \gamma(\eta^*)\beta_s \nu]|_{\epsilon=0^+}. $$

**Proof.** See the appendix, section G.6.

This decomposition provides an additional perspective on why contracts with low values of $\alpha$ end up “equity-like” in the best states of the world. For these cost functions, $\hat{p}(p(\eta^*))$ places low weight on the best states of the world. As a result, the increased effort that results from an alignment of the seller’s incentives and the asset value in those states (the covariance term in corollary 3) is small. The risk-shifting that occurs because the seller’s retained tranche does not resemble an equity claim (the variance term in corollary 3) in those states is also small. It is therefore efficient to give more of the cashflows to the buyer in the best states than it would be under the KL divergence cost function, because the gains from trade effects are larger than the moral hazard effects, and this results in an increasing, equity-like security design.

In the next section, I will show that these results— the optimality of a mixture of equity and debt for the alpha divergences and the notion of a mean-variance tradeoff for the security design problem— apply in an approximate sense to a much larger class of cost functions.

**5. Approximations**

In this section, I will discuss “approximately optimal” security designs. The approximation is motivated by the following observation: for the optimal security designs with an $\alpha$-divergence cost function (proposition 3), the slope of the security design depends on the gains from trade $\kappa$ and the parameter $\alpha$. In many applications, the percentage gains from trade might be quite small. For example, in the context of collateralized loan obligations, Nadauld and Weisbach [2012] estimates the cost of capital advantage due to securitization at 17 basis points per year. Assuming a five-year maturity, this would imply that the buyer’s valuation of the security is roughly one percent
higher than the seller valuation of the security. This finding accords with intuition—there are many economic forces (the availability of substitute securities that both the buyer and seller can trade, entry into the securitization business) that act to diminish differences in valuations.

Consider an example optimal security design: suppose the cost function is the $\chi^2$-divergence ($\alpha = -3$), and the gains from trade are one percent. In this case, the slope of the “equity portion” of the optimal contract is

$$\frac{-\kappa(1 + \alpha)}{2 + \kappa(1 - \alpha)} = \frac{0.02}{2 + 0.04} \approx 1\%.$$

The optimal contract is a debt plus a roughly one-percent equity claim for the buyer; intuitively, a standard debt contract cannot be substantially worse, from a welfare perspective. This argument used a specific cost function, but the point holds generally—unless the curvature of the cost function changes rapidly ($\alpha$ is very large or very small), the optimal security designs will resemble a debt contract.

This argument leads to a second observation: that in models with small gains from trade, which nevertheless result in a large quantity of trade, the moral hazard must also, in some sense, be small. Recall that we normalized the problem so that the expected value of the assets, if the seller retains everything, is one. Suppose that the moral hazard is large (e.g. that the expected value of the assets, if the seller retains nothing, is one-half). If the gains from trade are one percent, then no trade is much better than selling everything. Inevitably, the optimal security design in this case will be close to selling nothing. In the example of securitization, this is counter-factual; a substantial portion of the value of the underlying assets is sold in most securitizations.

This leads us to the conclusion that the moral hazard must also be small, in the sense that the difference in the seller’s effort between when she sells everything and when she sells nothing must be of similar magnitude to the gains from trade. In the case of securitization, this is consistent with empirical estimates (see the appendix, section §C). The smallness of the moral hazard means that poorly designed contracts cannot destroy entirely the value of the assets; however, they can destroy entirely the gains from trade. It does not mean that moral hazard is unimportant. In the calibration for mortgage securitization in the appendix, section §C, I find that using the “right” security design can substantially increase the profitability of securitization. We will see that, depending on the relative size of the moral hazard and gains from trade, no trade, trading everything, and many securities in between are consistent with both the moral hazard and gains from trade being small. In other words, the moral hazard can be small relative to the notional (asset) value being traded, but large relative to the profitability of trade, and the latter comparison will determine whether moral hazard impedes trade.

Formally, the approximations I consider are first- and second-order expansions of the utility function in the security design problem. I approximate the utility of using an arbitrary security
design $s$, relative to selling nothing, to first or second order in $\theta^{-1}$ and $\kappa$. When $\theta^{-1}$ is small, and therefore $\theta$ is large, it is difficult for the seller to change $p$. When $\kappa$ is small, the gains from trade are low. I take this approximation around the limit point $\theta^{-1} = \kappa = 0$. This approximation applies when $\theta^{-1}$ and $\kappa$ are small but positive, consistent with the arguments above. The limit point itself is degenerate; because there is no moral hazard and no gains from trade, the security design does not matter. However, near the limit point (where the approximation applies), this is not the case; some security designs are better than other security designs. The relevance of the approximation will depend on whether $\theta^{-1}$ and $\kappa$ are small enough, relative to the higher order terms of the utility function, for those terms to be negligible. This is a question that can only be answered in the context of a particular application. In the appendix, section §C, I discuss a calibration of the model relevant to mortgage origination, for which the approximation is accurate.

The results of this section apply to all invariant divergences, a class which includes all of the $f$-divergences, and therefore the KL divergence and the $\alpha$-divergences. This class also includes divergences, such as the Chernoff and Bhattacharyya distances, that are not additively separable. Using the approximation described above, I show that debt securities achieve, up to first order, the same utility as the optimal security design, for any invariant divergence cost function. Moreover, only debt contracts have this property, and it arises through the mean-variance intuition discussed in the previous section. I also show that the optimal contracts corresponding to the $\alpha$-divergences achieve, up to second order, the same utility as the optimal security design, for any invariant divergence cost function. This also follows from the mean-variance intuition discussed previously.

To further develop the intuition behind this result, consider the $f$-divergences. For any $f$-divergence, we can approximate the divergence to third order around $p = q$ as

$$\sum_{i \in \Omega} q_i f\left(\frac{p_i}{q_i}\right) \approx \sum_{i \in \Omega} q_i \left(\frac{1}{2} \left(\frac{p_i}{q_i} - 1\right)^2 - \frac{1}{12}(\alpha + 3)\left(\frac{p_i}{q_i} - 1\right)^3\right),$$

where we have defined $\alpha$ to satisfy

$$f'''(1) = -\frac{1}{2}(\alpha + 3).$$

This definition of $\alpha$ is consistent with the definition of the parameter $\alpha$ for the $\alpha$-divergences. This Taylor expansion shows that, up to third order, any $f$-divergence can be approximated by an $\alpha$-divergence. Intuitively, it follows that, for every $f$-divergence, the optimal contract associated with that $f$-divergence and the optimal contract for an $\alpha$-divergence (with $\alpha$ defined as above) will achieve approximately the same utility in the security design problem.

A different way to view the same results is through the lens of the perturbation argument employed in the previous section. The indirect effect of the perturbation described previously is
governed by the term
\[ \frac{p^j(\eta^*)}{q^j} f''\left(\frac{p^j(\eta^*)}{q^j}\right) - \frac{p^i(\eta^*)}{q^i} f''\left(\frac{p^i(\eta^*)}{q^i}\right). \]

Using the first-order condition in the moral hazard problem, one can observe that as \( \theta \) becomes large (\( \theta^{-1} \) small), holding the retained tranche \( \eta \) fixed, \( p(\eta) \) converges to \( q \). Intuitively, as it becomes increasing costly for the seller to keep \( p \) away from \( q \), she responds by moving \( p \) closer to \( q \). In the limit, \( p \) reaches \( q \), and the indirect effect of the utility perturbation is zero. With the KL divergence, the indirect effect is always zero. When the indirect effect is zero, the perturbation argument described in section §2 applies, and the optimal contract is debt. The proposition and corollary below make these arguments formally.

First, however, I will introduce a lemma, based on the results of Chentsov [1982]. This lemma can be thought of as showing that, up to third-order, all invariant divergences with continuous third derivatives are equivalent to an \( \alpha \)-divergence. That is, this lemma extends the Taylor approximation approach described above to divergences that are invariant but not additively separable.

**Lemma 2.** For every invariant divergence with continuous third derivatives, there exist a positive constant \( c \) and a real number \( \alpha \) such that

\[ \frac{\partial^2 D(p||q)}{\partial p^i \partial p^j} \bigg|_{p=q} = c \cdot g_{ij}(q), \]

where \( g_{ij}(q) \) is the Fisher information matrix, and

\[ \frac{\partial^3 D(p||q)}{\partial p^i \partial p^j \partial p^k} \bigg|_{p=q} = c\left(\frac{3 + \alpha}{2}\right) \partial_i g_{jk}(p) \bigg|_{p=q}. \]

**Proof.** See appendix, section G.7. The first claim is from Chentsov [1982], and the second follows from results found in that text. \( \square \)

The results of this lemma illustrate the connection between the mean-variance analysis discussed in the previous section and the approximation employed in this section. We can approximate the Hessian, in the neighborhood around \( q \), as

\[ \frac{\partial^2 D(p||q)}{\partial p^i \partial p^j} \approx c g_{ij}(\tilde{p}), \]

where

\[ \tilde{p} = q + \left(\frac{3 + \alpha}{2}\right) (p - q). \]

That is, \( \tilde{p} \) is a first-order approximation of the distorted probability \( \hat{p} \) described in the previous section (equation (4.2)). It plays the same role in the security design problem. If \( \tilde{p} = p \), the \( \alpha = -1 \) case, then the Hessian will be the Fisher information matrix at \( p \), which is the KL divergence case,
and debt will be approximately optimal. If $\alpha > -1$, $\tilde{p}$ will place more weight on the best states of the world, and the optimal contract will slope downward. If $\alpha < -1$, the reverse will be true.

The proposition below formalizes these results. As in previous sections, the proposition applies to small perturbations of the security design, in the neighborhood of the optimal security design. However, it is possible to characterize the utility in the security design problem up to second order, for all security designs, not just those that are close to the optimal security design. In fact, up to first order, the utility in the security design problem is exactly a mean-variance tradeoff, as proposition 11 in the appendix demonstrates.

I consider a third-order asymptotic expansion of the security design problem utility, $U(s; \theta^{-1}, \kappa)$, around the point $\theta^{-1} = \kappa = 0$, holding $\beta_s$ fixed as $\kappa$ changes:

**Proposition 4.** In the non-parametric model, with a smooth, convex, invariant divergence cost function, the effects of any security design perturbation (equation (2.3)) are, up to second order,

$$\frac{\partial U(\eta(\epsilon))}{\partial \epsilon} \bigg|_{\epsilon = 0^+} = \kappa \frac{\partial}{\partial \epsilon} E_{p^\eta}[\beta_s s(\epsilon)] \bigg|_{\epsilon = 0^+} - \frac{1}{2} (1 + \kappa) \theta^{-1} \frac{\partial}{\partial \epsilon} V_{p^\eta}[\beta_s s(\epsilon)] \bigg|_{\epsilon = 0^+} + O(\theta^3 + \kappa\theta^{-2}),$$

where

$$p^i(\eta) = q^i + \theta^{-1} q^i \cdot (\eta_i - \sum_{j \in \Omega} q^j \eta_j) + O(\theta^{-2})$$

and

$$\tilde{p}^i(p(\eta)) = q^i + \theta^{-1} \left( \frac{3 + \alpha}{2} \right) q^i \cdot (\eta_i - \sum_{j \in \Omega} q^j \eta_j) + O(\theta^{-2}).$$

To first order,

$$\frac{\partial U(\eta(\epsilon))}{\partial \epsilon} \bigg|_{\epsilon = 0^+} = \kappa \frac{\partial}{\partial \epsilon} E_{p}[\beta_s s(\epsilon)] \bigg|_{\epsilon = 0^+} - \frac{1}{2} \theta^{-1} \frac{\partial}{\partial \epsilon} V_{p}[\beta_s s(\epsilon)] \bigg|_{\epsilon = 0^+} + O(\theta^{-2} + \kappa\theta^{-1}).$$

**Proof.** See appendix, section G.9. \qed

The accuracy of the approximation that both the moral hazard and gains from trade are small will vary by application. The generality of proposition 4, which holds for all sample spaces, zero-cost distributions, and invariant divergences, suggests that as long as the moral hazard is not too large, the agents can neglect the details of the cost function.

The first-order and second-order results of proposition 4 are reminiscent of the mean-variance perturbation results described in the previous sections. In both cases, the direct effect is the change in the expected value under an fixed, possibly endogenous probability distribution, and the indirect effect is the change in the variance under another fixed, endogenous probability distribution. To first-order, and to second-order when $\alpha = -1$, the two probability distributions are the same. This
was also the case under the KL divergence, and as a result, debt securities are always first-order optimal, and second-order optimal when $\alpha = -1$. When $\alpha \neq -1$, the probability distributions are different, as in the general case of $\alpha$-divergences. In this case, the optimal security design for that $\alpha$-divergence will be the second-order optimal security design. The corollary below states these results formally.

**Corollary 4.** Under the assumptions of proposition 11, there exists a debt security, $s_{\text{debt}}$, for which the difference between the utility achieved by that security and the optimal security design, $s^*$, is second order:

$$U(s^*; \theta^{-1}, \kappa) - U(s_{\text{debt}}; \theta^{-1}, \kappa) = O(\theta^{-2} + \kappa \theta^{-1}).$$

Under those same assumptions, there exists a security, $s_{\text{debt-eq}}$, that is the optimal security design for an $\alpha$-divergence cost function (proposition 3), for which the difference between the utility achieved by that debt security and the optimal security design, $s^*$, is third order:

$$U(s^*; \theta^{-1}, \kappa) - U(s_{\text{debt-eq}}; \theta^{-1}, \kappa) = O(\theta^{-3} + \kappa \theta^{-2}).$$

**Proof.** See appendix, section G.10. □

The results for first-order and second-order optimal security designs can be summarized as a type of “pecking order” theory (when $\alpha \geq -1$). When the moral hazard and gains from trade are small, the agents can use debt contracts. As the stakes grow larger, so that both the moral hazard and gains from trade are bigger concerns, the agents can use a mix of debt and equity. For very large stakes, the security design will depend on the precise nature of the moral hazard problem.

The result of corollary 4 shows that when the gains from trade and moral hazard are small, but not zero, debt is approximately optimal in a way that other security designs are not. In the appendix, Figure A.3, I illustrate this idea. I assume an $\alpha$-divergence cost function, with $\alpha = -7$, which results in an optimal contract that is a mixture of debt and equity. I plot the utility of this optimal contract, as well as the best debt contract and best equity contract, relative to selling everything, for different values of $\theta$, with $\kappa = \bar{\kappa} \theta^{-1}$. As $\theta$ becomes large, all security designs converge to the same utility. For intermediate values of $\theta$, the best debt contract achieves nearly the same utility as the optimal contract, which is what the first-order approximation results show. For low values of $\theta$, the gap between the optimal debt contract and optimal contract grows.

It is important to emphasize that the securities described in corollary 4 are not degenerate; the debt security that is first-order optimal will not, in general, be selling everything or selling nothing. The level of the debt will be determined by the probability distribution $q$ and the product of $\kappa$ and $\theta$, as described in proposition 1. The approximation I have employed assumes that $\kappa$ is small and $\theta$ is large, but makes no assumption about their product. If the gains from trade are large relative to
the moral hazard (κθ large), the level of the debt will be high. If the moral hazard is large relative to the gains from trade (κθ small), the level of the debt will be small.

As in the previous sections, we can decompose the “indirect effects” of changing the security design, which are captured by the variance term in the mean-variance tradeoff described in proposition 4, into effort and risk-shifting components, as described by lemma 1. The corollary below formalizes the result.

**Corollary 5.** With an invariant divergence cost function, the indirect, effort-only effect of a security design perturbation (see lemma 1) is

\[
\frac{\beta_b}{\beta_s}(1 - \gamma(\eta^*)) \frac{d\epsilon}{d\epsilon} |_{\epsilon=0+} = \theta^{-1}(1 + \kappa)(1 - \gamma(\eta^*)) \frac{\partial}{\partial \epsilon} \text{Cov}(p(\eta^*))(\eta(\epsilon), \beta_s v) |_{\epsilon=0+} + O(\theta^{-3} + \kappa \theta^{-2}),
\]

where \( \tilde{p}(p(\eta^*)) \) is defined as in proposition 4. The indirect effect of a security design perturbation on risk-shifting is

\[
-\frac{\beta_b}{\beta_s} \sum_{j \in \Omega} \frac{dp^j(\eta(\epsilon))}{d\epsilon} \bigg|_{\epsilon=0+} = \frac{1}{2} \theta^{-1}(1 + \kappa) \frac{\partial}{\partial \epsilon} V(\tilde{p}(p(\eta^*))[\eta(\epsilon) - \gamma(\eta^*) \beta_s v]) |_{\epsilon=0+} + O(\theta^{-3} + \kappa \theta^{-2}).
\]

**Proof.** The corollary follows from proposition 4 and the proof of corollary 3.

The intuition discussed in the previous section holds. To first order, the effort and risk-shifting effects are the covariance and variance under the probability distribution \( q \). To second order, the relevant probability distribution is distorted, in a direction that depends on whether \( \alpha \) is greater than or less than negative one.

The exact and approximate results of the last two sections apply to non-parametric models, in which the seller can choose any distribution. In the appendix, sections D and E, I analyze parametric models using similar methods. In the next two sections of the paper, I will discuss continuous time models of effort. I will show that these models are essentially equivalent to the non-parametric models analyzed thus far. As a result, the optimality of debt and the intuitions about mean-variance tradeoffs apply in to these models as well. These sections can also be thought of as providing a micro-foundation for the static models discussed thus far.

**6. Dynamic Moral Hazard**

In this section, I will analyze a continuous time effort problem. This problem is closely connected to the static models discussed previously. The role of this section is to explain how an agent...
could “choose a distribution,” and show that the mean-variance intuition and optimality of debt discussed previously apply in dynamic models.

I will study models in which the seller controls the drift of a Brownian motion. The contracting models I discuss are similar to those found in Holmström and Milgrom [1987], Schaettler and Sung [1993], and DeMarzo and Sannikov [2006], among others. The models can be thought of as the continuous time limit of repeated effort models\textsuperscript{12}, in which the seller has an opportunity each period to improve the value of the asset. Two recent papers are particularly relevant. The models I discuss are a special case of Cvitanić et al. [2009]. I build on the results of Bierkens and Kappen [2014], who study a single-agent control problem (e.g. the seller’s moral hazard problem) with quadratic effort costs, and show that it is equivalent to a relative entropy minimization problem.

Relative these papers, I make two contributions. First, I show that the entire class of models studied by Cvitanić et al. [2009] can be rewritten as the static, non-parametric security design problems studied in the previous sections of the paper. That is, the dynamic models discussed in this section can be thought of as providing a micro-foundation for the static problems discussed in the previous sections. For the particular case of quadratic costs and a risk-neutral seller, the results of Cvitanić et al. [2009] imply that debt is optimal. Combining my result with the results of Bierkens and Kappen [2014], dynamic models with quadratic effort costs are equivalent to static problems with a KL divergence cost function, which provides a different perspective on why debt is optimal in this setting.

Second, I show that for convex, but not necessarily quadratic, cost functions, debt contracts are approximately optimal, and relate this to the mean-variance ideas discussed previously. This can be viewed as a micro-foundation for the approximation results discussed in the previous section. The result is also useful because the optimal contracts in this case are quite complex; Cvitanić et al. [2009] study contracts without the limited liability constraint, and show that they depend on the entire path, not just the final value, of the state variables. There are, to my knowledge, no known results with limited liability. My results can be viewed as showing that, when the approximation is applicable, simple, non-path-dependent contracts are close to optimal.

I will begin by describing the structure of the dynamic model. The timing follows the standard principal-agent convention. At time zero, the seller and buyer trade a security. Between times zero and one, the seller will apply effort (or not) to change the value of the asset. At time one, the asset value is determined and the security payoffs occur.

Between times zero and one, the seller controls the drift of a Brownian motion. Define $W$ as a Brownian motion on the canonical probability space, $(\Omega, \mathcal{F}, \tilde{P})$, and let $\mathcal{F}_t^W$ be the standard

\textsuperscript{12}See Biais et al. [2007], Hellwig and Schmidt [2002], Sadzik and Stacchetti [2015] for analysis of the relationship between discrete and continuous time models.
augmented filtration generated by $W$. Denote the asset value at time $t$ as $V_t$, and let $\mathcal{F}_t^V$ be the filtration generated by $V$. The seller observes the history of both $W_t$ and $V_t$ at each time, whereas the buyer observes (or can contract on) only the history of $V_t$. This information asymmetry creates the moral hazard problem.

The initial value, $V_0 > 0$, is known to both the buyer and the seller. The asset value evolves as

$$dV_t = b(V_t, t)dt + u_t \sigma(V_t, t)dt + \sigma(V_t, t)dW_t,$$

where $b(V_t, t)$ and $\sigma(V_t, t) > 0$ satisfy standard conditions to ensure that, conditional on $u_t = 0$ for all $t$, there is a unique, everywhere-positive solution to this SDE.\(^\text{13}\) The seller’s control, $u_t$, should be thought of as instantaneous effort (and not “effort” in the sense of the effort/risk-shifting decomposition discussed earlier). There is a flow cost of instantaneous effort, a general form of which is $g(t, V_t, u_t)$. The function $g(\cdot)$ is weakly positive, twice-differentiable, and strictly convex in instantaneous effort. For all $t$ and $V_t$, $g(t, V_t, 0) = 0$. Instantaneous effort always improves the expected value of the asset, holding future effort constant; that is, for all $t$ and $V_t$, $E_t[V_s]$ is increasing in $u_t$, for all $s > t$.

In the most general formulation possible, the seller’s information set at each time $t$ consists of the current time, the histories of the Brownian motion $W$ and asset value $V$, the history of her past actions, and any public or private randomization devices she chooses to employ. Using this information, the seller could pursue pure or mixed strategies over instantaneous effort levels. However, for the models that I will discuss, it is without loss of generality to restrict the seller to strategies that are a function of the history of the asset values and time (see Cvitanić et al. [2009]).

Intuitively, the convexity of the cost of instantaneous effort makes mixed strategies sub-optimal. Moreover, the security is a function of a first-order conditional expectation of the asset value, holding future effort constant; that is, for all $t$ and $V_t$, $E_t[V_s]$ is increasing in $u_t$, for all $s > t$.

The following conditions are sufficient. For all $V \in \mathbb{R}^+$ and $t \in [0, 1]$, $\sigma(V, t) > 0$ and $|b(V, t)| + |\sigma(V, t)| \leq C(1 + |V|)$ for some positive constant $C$. For all $t \in [0, 1], V, V' \in \mathbb{R}^+$, $|b(V, t) - b(V', t)| + |\sigma(V, t) - \sigma(V', t)| \leq D|V - V'|$, for some positive constant $D$. For all $t \in [0, 1]$, $\lim_{v \to 0^+} \sigma(v, t) = 0$, and $\lim_{v \to 0^+} b(t, v) \geq 0$.\(^\text{13}\)
\( \phi_{CT}(\eta) = \sup_{\{u_t\} \in \mathcal{U}} \phi_{CT}(\eta; \{u_t\}) = \sup_{\{u_t\} \in \mathcal{U}} \left\{ E^\hat{P}[\eta(V)] - E^\hat{P}\left[ \int_0^1 g(t, V_t, u_t) dt \right] \right\}, \)

where \( E^\hat{P} \) denotes the expectation at time zero under the physical probability measure.\(^{14}\) In summary, given the retained tranche, the seller chooses a time-consistent instantaneous effort strategy to control the drift of the asset value.

The security design problem is similar to the security design problem in the previous sections. The seller internalizes the effect that the security design will have on the price that the buyer is willing to pay, and solves

\[
U_{CT}(s^*) = \sup_{s \in S} U_{CT}(s) = \sup_{s \in S} \{ \beta_s E^\hat{P}[s(V)] + \phi_{CT}(\eta) \},
\]

where \( S \) is the set of \( \mathcal{F}_1^V \)-measurable limited liability security designs and \( \eta(V) = \beta_s(V_1 - s(V)) \).

In the proposition below, I show that this problem is equivalent to a static, non-parametric security design problem. Equivalent, in this context, means that the utility achieved by the seller in the continuous time problem, for any admissible security design, is equal to the utility achieved by that security in the static, non-parametric security design problem.

**Proposition 5.** There exists a probability space \((\Omega, \mathcal{F}, Q)\), Brownian motion \(B \) defined on that probability space, and stochastic process

\[
dX_t = b(X_t, t) dt + \sigma(X_t, t) dB_t,
\]

such that:

1. For all strategies \( u \in \mathcal{U} \), there exists a measure \( P \) under which the law of \( X \) is equal to the law of \( V \) under measure \( \hat{P} \).
2. For all securities \( s \in S \), the indirect utility function satisfies

\[
\phi_{CT}(\eta) = \sup_{P \in M} E^P[\eta(X)] - D_\gamma(P||Q),
\]

where \( D_\gamma \) is a divergence and \( M \) is the set of measures on the probability space that are absolutely continuous with respect to \( Q \) and for which \( E^Q[(dP/dQ)^4] < \infty \).

\(^{14}\)If the buyer and seller were risk-averse, but shared a common risk-neutral measure \( \hat{P} \), the problem would be identical. The key assumption in that case would be that the problem is small, in the sense that the outcome of this particular asset and security does not alter the common risk-neutral measure.
For all securities $s \in S$, if there is a unique maximizer $P(\eta) = \arg \max_{P \in M} E^P[\eta(X)] - D_g(P||Q)$, then security design utility function satisfies

$$U(s) = \beta E^{P(\eta)}[s(X)] + E^{P(\eta)}[\eta(X)] - D_g(P(\eta)||Q).$$

Proof. See appendix, section G.14. The proposition relies on Girsanov’s theorem and the “weak formulation” results of Schaettler and Sung [1993] and Cvitanić et al. [2009].

This proposition connects the dynamic problem introduced in this section to the static problems described in the previous sections. The intuition is that instantaneous effort strategies can be used to create any probability measure over outcomes, where an outcome is a path of the asset value. Given any point in time and history of the asset value, if the seller would like to make paths that move upward at this point more likely than paths that move downward, she can exert instantaneous effort. By doing this at each possible time and history, the seller can use her control to pick the relative likelihood of every path of the asset value. Formally, this intuition is captured by Girsanov’s theorem.

These results also show that the decomposition of the seller’s actions into “effort” and “risk-shifting”, as described by lemma 1, apply to these dynamic models as well. To prevent confusion, I will refer to the sort of effort described by lemma 1 as “cumulative effort,” and continue to use the term “instantaneous effort” to refer to the control the seller uses. The distinction between cumulative effort and instantaneous effort is related to another important point: even though the agent does not control the instantaneous variance of the asset value process, she can “spread out” the probability measure over asset value paths, creating risk-shifting effects.

The proof of the proposition shows that, for any measure $P$, there is a (stochastically) unique effort strategy that will create that measure. The divergence $D_g(P||Q)$ is the expected cumulative flow cost $g(\cdot)$ of this effort strategy. It satisfies the properties of a divergence– it is zero if $P$ is identical to $Q$, and positive otherwise. The measure $Q$ is the measure that corresponds to zero effort; if the agent exerts zero effort for all possible histories, the law of $X$ under measure $Q$ will be equal to the law of $V$ under measure $\tilde{P}$.

One technical caveat is included in the third part of the proposition. Thus far, I have not made enough assumptions about asset value process to ensure that there is a unique optimal measure, $P(\eta)$, or that the seller’s utility is finite. When I discuss specific cost functions $g$ below, I will introduce additional assumptions about the asset value process to ensure utility is finite and that there is a unique measure that solves the moral hazard problem.

I have rewritten the continuous time moral hazard problem as a static problem, in which the seller chooses a probability measure subject to a cost that is described by a divergence. In light of the results for static models, two questions immediately arise. First, is there a $g(\cdot)$ function such
that $D_g(P||Q)$ is the Kullback-Leibler divergence, in which case a debt security will be optimal? Second, are there $g(\cdot)$ functions such that $D_g(P||Q)$ is an invariant divergence, in which case a debt security will be approximately optimal?

The answer to the first question comes from the work of Bierkens and Kappen [2014] and the sources cited therein, who show that quadratic costs functions, $g(t, X_t, u_t) = \theta u_t^2$, lead to the KL divergence.\textsuperscript{15} Intuitively, it follows that the optimal security design is a debt security. This intuition is confirmed by specializing of the results of Cvitanic et al. [2009] to the case of a risk-neutral agent. For completeness, I present this result below, and include a proof in appendix. The proof also demonstrates that the decomposition of a perturbation’s effects into direct and indirect effects, and the further decomposition of the indirect effect into cumulative effort and risk-shifting effects, discussed in previous sections, apply to these models as well.

For the quadratic flow cost function, it is sufficient to assume that the asset value, in the absence of effort by the agent, satisfies $E_Q[\exp(4 \theta - 1 X_1)] < \infty$, which ensures that utility is finite and that there is a unique optimal policy for the seller.

**Proposition 6.** In the continuous time model, with the quadratic cost function, if $E_Q[\exp(4 \theta^{-1} X_1)] < \infty$, the optimal security design is a debt contract,

$$s(X) = \min(X, \bar{v}),$$

for some $\bar{v} > 0$.

The decomposition of perturbations into direct and indirect effects (equation (2.3)) applies:

$$\left. \frac{\partial U_{CT}(\eta(X, \epsilon))}{\partial \epsilon} \right|_{\epsilon = 0} = \kappa \frac{\partial}{\partial \epsilon} E^{P^*(\eta^*)}[\beta_s s(X, \epsilon)] - \left(1 + \kappa\right) \frac{1}{2} \theta^{-1} \frac{\partial}{\partial \epsilon} V^{P^*(\eta^*)}[\beta_s s(X, \epsilon)].$$

The effort/risk-shifting decomposition also applies:

$$\left. \frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{d(\eta(\epsilon))}{d\epsilon} \right|_{\epsilon = 0^+} = \theta^{-1} \frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{\partial}{\partial \epsilon} \text{Cov}(P^*(\eta^*)|\eta(\epsilon), \beta_s X)|_{\epsilon = 0^+}$$

and

$$\left. \frac{\beta_b}{\beta_s} \frac{d}{d\epsilon} E^{P(\eta(\epsilon))}[\eta^*(X) - \gamma(\eta^*)\beta_s X] \right|_{\epsilon = 0^+} = \frac{1}{2} \theta^{-1} \frac{\beta_b}{\beta_s} \frac{\partial}{\partial \epsilon} V^{P(\eta^*)}[\eta(\epsilon) - \gamma(\eta^*)\beta_s X]|_{\epsilon = 0^+}.$$

**Proof.** See appendix, section G.15. The result on the optimality of debt is a special case of Cvitanic et al. [2009].\qed

\textsuperscript{15}Note that this formulation rules out time discounting of the effort costs. One way to motivate this assumption is to suppose that neither agent discounts the future, but the seller is required to raise $I$ dollars to initiate the project. In this case, the gains from trade is really just the multiplier on this constraint, rather than any difference in patience between the two agents.
Debt is the optimal security design in the continuous time model for same reasons it is optimal in the non-parametric model. The perturbation argument used in section §3 applies without material modification. The mean-variance perturbation argument also applies; there is an infinite-dimensional version of the Fisher information “matrix,” which I will discuss in the next section, that allows the argument to proceed.

The intersection of these results with Holmström and Milgrom [1987] is intuitive. In the principal-agent framework, when the asset value Ito process is an arithmetic Brownian motion and the flow cost function is quadratic, without limited liability, a constant security for the principal is optimal. With limited liability, in the security design framework, optimal security simply reduces the constant payoff where necessary, and debt is optimal.

The debt security design may or may not be renegotiation-proof. Suppose that at some point, say time \( t = \frac{1}{2} \), the seller can offer the buyer a restructured security. Assume that at this time, there are no gains from trade (otherwise, if the asset value has increased, the seller will “lever up” and sell more debt to the buyer). If the current asset value is low enough, the debt security provides little incentive for the seller to continue putting in effort in the future. In this state, the buyer might agree to “write down” the debt security, even though he cannot receive any additional payments from the seller, because the buyer’s gains from increased effort by the seller could more than offset the loss of potential cash flows. In this model, write-downs can be Pareto-efficient if the time-zero expected value of the debt, \( E^P[s(X)] \), is greater than \( \theta \). Write-downs will never be Pareto-efficient when \( \kappa \) and \( \theta^{-1} \) are both small, but could occur if both the gains from trade at time zero and the moral hazard were large.

In the next section, I turn to the second question: are there cost functions \( g(\cdot) \) for which debt securities are approximately optimal?

7. A Mean-Variance Approximation for Continuous Time Models

For the static models discussed earlier, invariant divergence cost functions lead to models in which debt was approximately optimal. The key mathematical property of invariant divergences that generated this result is the fact that their Hessian matrices, evaluated at \( p = q \), are proportional to the Fisher information matrix.

In this section, I will not directly answer the question of whether there functions \( g(\cdot) \) such that \( D_g(P||Q) \) is invariant. Instead, I will show that for all \( g(t, X_t, u_t) = \theta \psi(u_t) \), where \( \psi(u_t) \) is a convex function, debt is approximately optimal. These flow cost functions will generate divergences \( D_g(P||Q) \) that, like the KL divergence, have the property that their “second variations” are

16This condition is sufficient, not necessary. I have omitted the proof for brevity.
MORAL HAZARD AND THE OPTIMALITY OF DEBT

proportional to the Fisher information. This is the infinite-dimensional analog of the property of invariant divergences mentioned above that leads to the approximate optimality of debt.

The approximations used in this section are identical to the ones discussed previously, in section §5. I consider problems in which both the moral hazard and gains from trade are small, relative to the scale of the assets. I show that the utility of arbitrary security designs can be characterized, to first-order, by a mean-variance tradeoff.

The approximate optimality of debt is a surprising result in this setting. Without limited liability, Cvitanić et al. [2009] are able to characterize some properties optimal security designs, making an analogy to the results of Holmström and Milgrom [1987]. However, there is no explicit solution or implementation available, and in general the optimal securities will be dependent on the entire path of asset values, not just the final value, in a non-trivial way. There are no results, to my knowledge, about the model with limited liability.

I modify the models introduced in the previous section in several small ways. I will assume that the control is bounded, \(|u_t| \leq \bar{u}\) (this is a restriction on the set \(\mathcal{U}\)). This assumption simplifies the discussion of conditions to ensure finite utility. I assume \(\psi\) satisfies the conditions required for \(g\) in the previous section, and in addition that for all \(|u| \leq \bar{u}\), \(\psi''(u) \in [K_1, K_2]\) for some positive constants \(0 < K_1 < 1 < K_2\). That is, \(\psi\) is “strongly convex” over its domain. I also normalize \(\psi''(0) = 1\). I assume that, for bounded control strategies \(|u| \leq \bar{u}\), the asset value has a finite fourth moment. That is, \(E_{\tilde{P}}[(V_1)^4] < \infty\) under these bounded control strategies.

There is a sense in which any twice-differentiable, convex cost function \(\psi(u_t)\) resembles the quadratic cost function, as \(u_t\) becomes close to zero, because their second derivatives are the same. Similarly, in static models, all invariant divergences resemble the KL divergence, because their Hessian is the Fisher information matrix. I apply this idea to the divergences \(D_\psi\) induced by the convex cost functions \(\psi\).

First, I define the Fisher information. Consider a model with a finite number of parameters, \(\tau\), and associated probability distribution \(p(\omega; \tau)\), over sample space \(\Omega\). The Fisher information matrix defined as

\[
I_{ij} = E_{p(\omega; \tau)}[(\frac{\partial \ln p(\omega; \tau)}{\partial \tau_i})(\frac{\partial \ln p(\omega; \tau)}{\partial \tau_j})].
\]

I will define the Fisher information for the continuous time model, which has an infinite number of parameters, in an analogous way. Let \(\frac{d\tilde{P}}{dQ}(\gamma, \tau) = \exp(\int_0^1 (\gamma u_s + \tau v_s) dB_s - \frac{1}{2} \int_0^1 (\gamma u_s + \tau v_s)^2 ds)\), where \(\gamma\) and \(\tau\) parametrize a perturbation in the direction defined by the square-integrable, predictable processes \(u_s\) and \(v_s\).\(^{17}\) The Fisher information, in the directions defined by \(u_s\) and \(v_s\), is

\(^{17}\)These perturbations are the Cameron-Martin directions used in Malliavin calculus.
defined as

\[ I(u, v) = E^{P(\gamma, \tau)}[(\frac{\partial}{\partial \gamma} \ln(dP_{\gamma, \tau}))(\frac{\partial}{\partial \tau} \ln(dP_{\gamma, \tau}))]|_{\gamma=\tau=0}. \]

Plugging in the definition of \( \frac{dP}{dQ}(\gamma, \tau) \), and using the Ito isometry, \( I(u, v) = E^{Q}[\int_{0}^{1} u_s v_s ds] \). Next, consider the second variation of \( D_\psi(P(\gamma, \tau)||Q) \),

\[ \frac{\partial}{\partial \tau} \frac{\partial}{\partial \gamma} D_\psi(P(\gamma, \tau)||Q)|_{\gamma=\tau=0} = \theta \frac{\partial}{\partial \tau} \frac{\partial}{\partial \gamma} E^{P(\gamma, \tau)}[\int_{0}^{1} \psi(\gamma u_s + \tau v_s)ds]|_{\gamma=\tau=0} = \theta E^{Q}[\int_{0}^{1} \psi''(0) u_s v_s ds]. \]

It follows that \( \frac{\partial}{\partial \tau} \frac{\partial}{\partial \gamma} D_\psi(P(\gamma, \tau)||Q)|_{\gamma=\tau=0} = \theta I(u, v) \). For any cost function \( \psi \), the second variation in the directions \( u_s \) and \( v_s \) is equal to the Fisher information in those directions. The divergences \( D_\psi \) resemble, locally, the KL divergence, in exactly the same way that all invariant divergences resemble the KL divergence.

I consider the same approximation discussed earlier, in which both \( \theta^{-1} \) and \( \kappa \) are small. In the context of continuous time models, Sannikov [2013] discusses a related “large firm limit.” As the cost of effort rises, the seller will choose to respond less and less to the incentives provided by the retained tranche. Regardless of the cost function \( \psi \), the divergence \( D_\psi(P||Q) \) will approach \( D_{KL}(P||Q) \), and debt will be approximately optimal. Moreover, the distinction between the effort and risk-shifting components of utility that applied in the static approximations will apply to these models as well. To make this argument rigorous, I use Malliavin calculus in a manner similar to Monoyios [2013] to prove the following theorem:

**Proposition 7.** For any limited liability security design \( s \), the difference in utilities achieved by an arbitrary security \( s \) and the sell-nothing security is

\[ U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = \kappa E^{Q}[\beta_s s] - \theta^{-1} \frac{1}{2} V^{Q}[\beta_s s] + O(\theta^{-2} + \theta^{-1} \kappa). \]

The direct and indirect effects of a perturbation, to first order, are identical to those described in proposition 6, under measure \( Q \). The decomposition of the indirect effect into effort-only and risk-shifting effects is also, to first-order, identical to the one described in proposition 6, under measure \( Q \).

**Proof.** See appendix, section G.16. □

In the continuous time effort problem with an arbitrary convex cost function, debt securities are first-order optimal. The same mean-variance intuition that I discussed in static models applies to continuous time models. The variance of the security payoff is again a summary statistic for the problems of reduced effort and risk shifting associated with the moral hazard problem.
8. Conclusion

In this paper, I have analyzed a flexible form of moral hazard, which allows for both effort and risk-shifting. In my benchmark model, with the KL divergence cost function, debt securities are exactly optimal. I provide a micro-foundation for this model in terms of a dynamic contracting problem with quadratic costs of effort. Other security designs (in some cases, a mix of debt and equity) are exactly optimal with the $\alpha$-divergence cost functions, and approximately optimal for the larger class of invariant divergence cost functions. In all of these models, debt is optimal or approximately optimal because it minimizes the variance of the security payout, balancing the need to provide incentives for effort, minimize risk-shifting, and maximize trade.

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FIGURE A.1. POSSIBLE SECURITY DESIGNS. This figure illustrates several possible security designs: a debt security, an equity security, and the “live-or-die” security of Innes [1990]. The x-axis, labeled $\beta_s v_i$, is the discounted value of the asset, and the y-axis, labeled $\beta_s s_i$, is the discounted value of the security. The level of debt, the cutoff point for the live-or-die, and the fraction of equity are chosen for illustrative purposes. The discount factor for the seller is $\beta_s = 0.5$. The outcome space $v_i$ is a set of 401 evenly-spaced values ranging from zero to 8. The x-axis is truncated to make the chart clearer.
FIGURE A.2. SECOND-ORDER OPTIMAL SECURITY DESIGNS. This figure shows the second-order optimal security designs, for various values of the curvature parameter $\alpha$. The x-axis, labeled $\beta_s v_i$, is the discounted value of the asset, and the y-axis, labeled $\beta_s s_i$, is the discounted value of the security. These securities are plotted with the same $\bar{v}$ for each $\alpha$ (not an optimal $\bar{v}$). The value of $\kappa$ used to generate this figure is one-third, which was chosen to ensure that the slopes of the contracts would be visually distinct (and not because it is economically reasonable). The outcome space $v$ is a set of 401 evenly-spaced values ranging from zero to 8.
Figure A.3. The utility of various security designs. This figure compares the utility of several security designs (debt, equity, and the optimal security design) relative to the utility of selling everything, for different values of θ. The bottom x-axis is the value of ln(θ), the top x-axis is the value of κ, and the y-axis is the difference in security design utility between the security (debt, equity, etc.) and selling everything. For each θ and corresponding κ, the optimal debt security, equity security, and the optimal security are determined. Then, the utility of using each of the four securities designs, given θ and κ, is computed. The cost function is a α-divergence, with α = −7, implying that a mix of debt and equity is optimal (see proposition 3). The gains from trade, κ, vary as θ changes, with κ = κθ⁻¹, κ = 0.0171. This parameter was chosen to be consistent with the calibration in the appendix, section §C. The discounting parameter for the seller is βₘ = 0.5. The zero-cost distribution q is a discretized, truncated gamma distribution with mean 2, 0.3 standard-deviation, and an upper bound of 8. The outcome space v is a set of 401 evenly-spaced values ranging from zero to 8. The utilities are plotted for nine different values of θ, ranging from 2 exp(−7) to 2 exp(1), and linearly interpolated between those values.
In this appendix section, I will discuss several possible timing conventions for the sequence of decisions by the seller during the first period. In that period, the seller designs the security, sells it to the buyer (assuming the buyer accepts), and takes actions that will create or modify the assets backing the security. The timing convention refers to the order in which these three steps occur. In the first timing convention, the “shelf registration” convention (using the terminology of DeMarzo and Duffie [1999]), the security is designed before the assets are created, but sold afterward. In the second timing convention, the “origination” convention, the security is designed and sold after the assets are created. In the third timing convention, the “principal-agent” convention, the security is designed and sold before the seller takes her actions. In this last convention, it is natural to assume that the asset exists before the security is designed, but its payoffs are modified by the seller’s actions after the security is traded. For the “principal-agent” timing convention, I will also discuss the effects of Nash-bargaining of the security price, and over both the security design and the security price.

There are asset securitization examples for each of these timing conventions. For some asset classes, such as first-lien mortgages, the security design is standardized, and the “shelf registration” timing convention is appropriate. For more unusual assets, the security design varies deal-by-deal, and the “origination” timing convention is appropriate. In some cases, such as the “Bowie bonds” (securitizations of music royalties), maintaining incentives post-securitization is important, and the principal-agent timing convention applies.

<table>
<thead>
<tr>
<th>Principal-Agent Timing</th>
<th>Origination Timing</th>
<th>Shelf Registration Timing</th>
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<tbody>
<tr>
<td>(1) Security Designed</td>
<td>(1) Actions Taken</td>
<td>(1) Security Designed</td>
</tr>
<tr>
<td>(2) Security Traded</td>
<td>(2) Security Designed</td>
<td>(2) Actions Taken</td>
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<tr>
<td>(3) Actions Taken</td>
<td>(3) Security Traded</td>
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The principal-agent timing convention is the simplest convention to analyze. In any sub-game perfect equilibrium, the seller takes actions that maximize the value of her retained tranche, because the price that she receives for the security has already been set. The buyer anticipates this, forming beliefs about the distribution of outcomes based on the design of the security. The buyer’s beliefs affect the price that he is willing to pay for the security, and the seller internalizes this when designing the security. Multiple equilibria are possible if the seller’s optimal actions for a particular retained tranche are not unique, or if there are multiple security designs that maximize the seller’s
utility. The moral hazard, in this timing convention, can occur either because the buyer is unaware of the seller’s actions, or because he can observe those actions but is powerless to enforce any consequences based on them.

Under the other two timing conventions, I use equilibrium refinements to argue that the optimal security design and actions associated with the principal-agent timing convention describe the most appealing equilibria of the game with those alternative timing conventions. I have drawn extensive-form game trees for these two timings in Figure B.1 and Figure B.2 below. The results I present are related to the findings of Matthews [1995] and Matthews [2001]. Matthews [1995] shows, in a closely related model in which contracts are renegotiable, and there is no limited liability, that all equilibria are “second best efficient,” which is related to my result that the timing is irrelevant. Matthews [1995] extends the results of Matthews [1995] to a model with limited liability, but with only one choice (effort) for the agent.

I assume that the actions of the seller are not observed by the buyer, ensuring there is still a moral hazard. I will discuss the benchmark, non-parametric model described in section §2; the set of feasible actions by the seller, \( M \), is the entire probability simplex. I use the notion of proper equilibrium defined by Myerson [1978], and developed for infinite action spaces by Simon and Stinchcombe [1995]. I show that, if the principal-agent timing convention has a unique equilibrium security design, price, and set of actions taken by the seller, which involve acceptance with certainty by the buyer, then this security design, price, set of actions, and acceptance with certainty also characterize all strong proper equilibria\(^\text{18}\) of the game with the origination and shelf registration timing conventions, subject to a technical assumption.

\(^{18}\)I have not shown that there is a unique strong proper equilibrium— in theory, there could be multiple equilibria with different acceptance strategies by the buyer for security/price combinations that never occur in equilibrium. However, results in the proof of proposition 8 lead me to believe this is not the case.
The key intuitions behind this result are the notions of “forward induction” (Kohlberg and Mertens [1986]) and “incredible beliefs” (Cho [1987]). Suppose that there is an equilibrium in
the origination timing in which the buyer is always offered a particular security, \( \bar{s} \). Now imagine that the seller plays an off-equilibrium strategy, and offers the buyer a different security, \( \hat{s} \). What should the buyer believe about the unobservable actions taken by the seller? The notion of forward induction recognizes the seller controls both the security design and her actions, and infers from the seller’s offer of security \( \hat{s} \) that the seller has taken actions consistent with the buyer accepting or rejecting that offer. That is, the seller might have taken actions that anticipated a lower or higher probability of the buyer accepting her offer, but the seller did not take actions that are not best responses to some acceptance strategy of the buyer, conditional on having offered the security \( \hat{s} \) to the buyer. As a result, the buyer should accept or reject the security \( \hat{s} \) based on the belief that the seller has acted in this way, and not rely on “incredible beliefs.” In particular, these notions rule out the idea that the buyer, when offered security \( \hat{s} \) instead of the security \( \bar{s} \), can believe the seller is “out to get him,” in the sense that the seller took actions that reduced her own utility in order to harm the buyer.\(^{19}\) These beliefs are not credible; the buyer cannot pretend to hold these beliefs in order to force the seller to offer him \( \bar{s} \) instead of \( \hat{s} \).

The notions of forward induction and incredible beliefs, and their associated refinements, are not generally equivalent to the proper equilibrium concept. The proper equilibrium concept imposes the constraint that, in the sequence of mixed strategies whose limit is the equilibrium, actions that result in greater utility for the seller must be more likely than actions resulting in lower utility for the seller. The buyer’s beliefs, which are governed by Bayes’ rule, must place relatively high weight on the seller playing best-response actions. As a result, in the game I study, proper equilibrium, forward induction, and restrictions against incredible beliefs end up implementing the same idea: that, off the equilibrium path, the buyer cannot believe the seller has played an action that is not a best response, conditional on her observable choice of security design.

The game is structured so that, if the buyer rejects the seller’s take-it-or-leave-it offer, the seller retains the entire asset (both the security and the retained tranche). For each security design, there is a one-dimensional manifold of best-response actions, each corresponding to a probability that the seller assigns to the buyer’s likelihood of acceptance. The worst case action in this one-dimensional manifold, from the perspective increasing the security’s value, is the action that corresponds to the seller believing the buyer will accept the security with certainty. In that case, the seller has no incentive to raise the value of the security. The other actions in this one-dimensional manifold correspond to best-responses in which the seller believes she might retain the security, and therefore acts to increase its value.\(^{20}\)

\(^{19}\)This argument uses the strategic form of the game, not the agent-strategic form (see Fudenberg and Tirole [1991], chapter 8.4). That is, off-equilibrium security designs are assumed to be correlated with off-equilibrium actions by the seller.

\(^{20}\)This result depends on the convexity of the set \( M \).
Now consider the optimal security design and price from the principal-agent timing. If the buyer is offered this security design and price, he must be weakly willing to accept, because regardless of the probabilities he assigns to the seller’s actions over this one-dimensional manifold, the price is at least fair. The seller, recognizing that the buyer will accept this security and price\textsuperscript{21}, must offer it— it maximizes her utility. This is a heuristic argument that outlines the proof in proposition 8, as it applies to the origination timing. I will now discuss the shelf registration timing, and then discuss the technical assumptions required by the proof.

In the shelf registration timing, the security is designed before the actions are taken. As a result, one might appeal to a notion of sequential rationality to capture the idea that the seller would not play non-best-response actions, conditional on the security design that has already been decided. However, the concept of sequential equilibrium is difficult to extend to games with infinite action spaces (see Myerson and Reny [2015]). I will instead use the proper equilibrium concept, recognizing that the results for the shelf registration timing might hold under a weaker equilibrium refinement.

There is also a significant technical assumption required for the proof of proposition 8. The technicality concerns the compactness of the action spaces available to the agents. The proof of proposition 8 relies on the proof of existence of strong proper equilibria (theorem 3.1) in Simon and Stinchcombe [1995], which itself requires that the action spaces of the agents be compact. This is problematic, because the buyer’s action space is the set of functions $A_b : S \times \mathbb{R} \rightarrow \{0, 1\}$, where $S$ is the set of limited liability securities, 0 represents rejection, and 1 represents acceptance of the offered security and price. This is not a compact space; the buyer could (in theory) accept some particular security and price, while rejecting every offer of the same security with a price arbitrarily close to the price the buyer would have accepted. The potential for this type of strategy leads Simon and Stinchcombe [1995] to require compact action spaces.

To circumvent these issues, I will require that the seller choose a security and price from a finite action space.\textsuperscript{22} That is, I will define the set $S$ of feasible security designs to be a finite set of possible security designs, all of which satisfy the limited liability constraints. I will define the set $K$ to be a finite set of feasible prices.

First, consider the principal-agent timing. Let $a(s, k)$ be the buyer’s acceptance strategy. The buyer must accept if the price, $k$, is less than the buyer’s valuation, $\beta_b \sum_{i>0} p^i(\eta(s)) s_i$, reject if the price is greater, and is indifferent if the price is equal to the buyer’s valuation. The seller’s payoff,

\textsuperscript{21}Actually, some price slightly lower but arbitrarily close to this price.

\textsuperscript{22}There are at least three possible alternative strategies. I could have required that the buyer’s strategy satisfy enough conditions to ensure compactness. Alternatively, I could have pursued the “limit-of-finite” approach described in Simon and Stinchcombe [1995]. Finally, I could have attempted to explicitly construct the sequence of mixed strategies that generate the proper equilibrium. Each of these seemed to require significant technical work that is beyond the scope of this paper.
given a particular acceptance strategy, is

\[ U(s, k; a) = (1 - a(s, k))\phi(\beta_s v) + a(s, k)(k + \phi(\eta(s))). \]

I assume that there is a unique sub-game perfect equilibrium in the principal-agent timing, and that this equilibrium involves acceptance by the buyer with certainty. Let the \( s^* \) and \( k^* \) denote the security design and price in this equilibrium, and let \( p^* = p(\eta(s^*)) \) denote the corresponding optimal actions. I also assume that the security \( s^* \) is not sell-nothing. I show that, under these assumptions, all strong proper equilibria of the games with the origination and shelf registration timing conventions are also characterized by the security design \( s^* \), the price \( k^* \), the action \( p^* \), and acceptance with certainty.

**Proposition 8.** In the non-parametric benchmark model described in section §2, if there is a unique sub-game perfect equilibrium for the game with the principal-agent timing convention, characterized by security design \( s^* \in S \), price \( k^* \in K \), actions \( p^* \in M \), and acceptance by the buyer, with \( s_i^* > 0 \) for some \( i \in \Omega \), then all strong proper equilibrium (in the terminology of Simon and Stinchcombe [1995]) of the origination timing and shelf registration timing are characterized by that security design, price, and action, and the buyer accepting the seller’s offer with certainty.

**Proof.** See appendix, section G.18.

The proposition argues that the timing of the game is, in essence, irrelevant. The analysis in the main body of the paper, regarding when debt contracts are optimal or nearly optimal, applies regardless of the timing. The proposition, as stated, relies on the strong proper equilibrium concept defined by Simon and Stinchcombe [1995], but also applies to those authors’ weak proper equilibrium concept.

Finally, I will discuss, under the principal-agent timing convention, alternatives to giving all of the bargaining power to the seller. I will discuss two alternatives: first, that the seller designs the security, but then Nash-bargains with the buyer over the price, and second, that the seller and buyer bargain jointly over both the security design and price.

First, suppose that the seller and buyer bargain over the price \( K(\eta) \). Let \( 1 - \rho > 0 \) and \( \rho > 0 \) be their respective bargaining weights. The outside option is no trade: the seller retains everything, and the buyer pays and receives nothing. The price, as a function of the retained tranche (or, equivalently, of the security design), solves

\[ K^*(\eta) \in \arg \max_K (\beta_b E^{p(\eta)}[s(\eta)] - K)^\rho(\phi(\eta) + K - \phi(\beta_s v))^{1-\rho}. \]

Using the first-order conditions to solve for \( K^*(\eta) \),

\[ (B.1) \quad K^*(\eta) = (1 - \rho)\beta_b E^{p(\eta)}[s] + \rho(\phi(\beta_s v) - \phi(\eta)). \]
The utility in the security design problem is
\[ U(\eta) = (1 - \rho)(\beta_b E^{p(\eta)}[s(\eta)] + \phi(\eta)) + \rho\phi(\beta_s v). \]

This is simply an affine transformation of the security design utility function described in the text (equation (2.2)), and it follows that the same security design will be optimal. The bargaining power, in this case, changes only the price at which the agents trade the security. Note also that, if the buyer (instead of the seller) designs the security, and then the agents bargain over the price, a similar result follows.

Now suppose that the agents bargain jointly over the security design and price. The agents maximize
\[ U(s^*) = \max_{K,s \in S} (\beta_b E^{p(\eta(s))[s]} - K)^\rho(\phi(\eta(s)) + K - \phi(\beta_s v))^{1-\rho}. \]

The optimal price, as a function of the optimal security design, is still described by equation (B.1). Substituting this in,
\[ U(s^*) = \max_{s \in S} (1 - \rho)^{1-\rho} \rho(\beta_b E^{p(\eta(s))[s]} + \phi(\eta(s)) - \phi(\beta_s v)), \]

which is also an affine transformation of the models described in the main text. It again follows that, if the agents bargain jointly over both the security design and price, the same security designs would be optimal.

APPENDIX C. CALIBRATION

In this section of the appendix, I will discuss possible calibration strategies for the static, non-parametric model of moral hazard discussed in the main text. I will focus on the context of mortgage securitization, and how to calibrate the key parameters \( \kappa \) and \( \theta \), under the assumption that the cost function is the KL divergence, or that the cost function is an invariant divergence and the first-order approximation discussed in the text is accurate. In both of these cases, a debt security design is optimal.

In the context of mortgage origination, there is empirical evidence for lax screening by originators who intended to securitize their mortgage loans, which suggests that moral hazard is a relevant issue (see Demiroglu and James [2012], Elul [2015], Jiang et al. [2013], Keys et al. [2010], Krainer and Laderman [2014], Mian and Sufi [2009], Nadauld and Sherlund [2013], Purnanandam [2011], Rajan et al. [2010], although some of this evidence is disputed by Bubb and Kaufman [2014]). However, some of this evidence is consistent with information asymmetries but cannot distinguish between moral hazard and adverse selection. There are also mechanisms to mitigate adverse selection by the seller, such as the inability to retain loans and random selection of loans into securitization (Keys et al. [2010]).
I will discuss an “experimental” approach to calibration first. This approach is consistent in spirit with the empirical literature on moral hazard in mortgage lending (Keys et al. [2010], Purnanandam [2011], others). In that literature, the quasi-experiment compares no securitization ($\eta_i = \beta s v_i$) with securitization. If we assume securitization uses the optimal security design $\eta^*$, then $\theta$ can be approximated (for any invariant divergence cost function, see section §5) as

$$\theta^{-1} \approx E^{p(\beta, v)}[v_i] \cdot \frac{E^{p(\beta, v)}[v_i] - E^{p^*}[v_i]}{Cov^{p^*}(v_i, s_i^*)}.$$  

This formula illustrates the difficulties of calibrating the model using the empirical work on moral hazard in mortgage lending. For the purposes of the model, what matters is the loss in expected value due to securitization, relative to the risk taken on by the buyers, ex-ante. The empirical literature estimates ex-post differences, and the magnitude of these differences varies substantially, depending on whether the data sample is from before or during the recent crash in home prices. Converting this into an ex-ante difference would require assigning beliefs to the buyer and seller about the likelihood of a crash. Estimating the ex-ante covariance, which can be understood as a measure of the quantity of “skin in the game,” is even more fraught. For these reasons, I have not pursued this calibration strategy further.

The second calibration strategy, which is somewhat more promising, is to use the design of mortgage securities to infer $\theta$. Essentially, by (crudely) estimating the other terms in the “put option value” equation (equation (3.2)), and assuming the model is correct, we can infer what the security designers thought the moral hazard was. Rearranging that equation,

$$\frac{\beta_b \bar{v} - \beta_b E^{p^*}[s_i]}{\beta_b E^{p^*}[s_i]} \frac{E^{p^*}[s_i]}{E^{p^*}[v_i]} (1 - \frac{E^{p(\beta, v)}[v_i] - E^{p^*}[v_i]}{E^{p(\beta, v)}[v_i]}) = \theta.$$  

The spread term should be thought of as reflecting the initial spread between the assets purchased by the buyer and the discount rate, under the assumption that the bonds will not default. Using a 90/10 weighting on the initial AAA and BBB 06-2 ABX coupons reported in Gorton [2008], I estimate this as 34 basis points per year. In a different setting (CLOs), the work of Nadauld and Weisbach [2012] estimates the cost of capital advantage (gains from trade) due to securitization at 17 basis points per year. The “share” term is the ratio of the initial market value of the security to the initial market value of the assets. Begley and Purnanandam [2014] document that the value of the non-equity tranches was roughly 99% of the principal value in their sample of residential mortgage securitizations. Similarly, the moral hazard term is likely to be small. The estimates of Keys et al. [2010], whose interpretation is disputed by Bubb and Kaufman [2014], imply that
pre-crisis, securitized mortgage loans defaulted at a 3% higher rate\textsuperscript{23} than loans held in portfolio. Assuming a 50% recovery rate, and using this as an estimate of the ex-ante expected difference in asset value, this suggests that the moral hazard term is roughly 1.5%, and therefore negligible in this calibration. Combining all of these estimates, I find $\theta$ of 2 is consistent with the empirical literature on securitization. This calibration assumed that the security design problem with the KL divergence was being solved. However, this formula also holds (approximately) under invariant divergences, conditional on the assumption that $\theta^{-1}$ and $\kappa$ are small enough.

The value of $\theta = 2$ can be compared with the results of Figure A.3. Under the assumptions used to generate that figure, which are described in its caption, I find that with $\theta = 2$ and $\kappa = 0.85$ (17 basis points per year times 5 years), debt would be achieve 99.96% of gains achieved by the optimal contract, relative to selling everything (and an even larger fraction of the gains relative to selling nothing). Under these parameters, the utility difference between the best debt security and selling nothing would be roughly 0.73% of the total asset value. While that might seem like an economically small gain, for a single deal described in Gorton [2008], SAIL 2005-6, the private gains of securitization would be roughly $16.4mm. In contrast, the utility difference between the best equity security and selling nothing is about 0.56% of the total asset value. The private cost of using the optimal equity contract, instead of the optimal debt contract, would be roughly $4mm for this particular securitization deal.

The numbers discussed in this calculation depend on the assumptions used in Figure A.3, some of which are ad hoc. Nevertheless, they illustrate the general point that it is simultaneously possible for debt to be approximately optimal, and for the private gains of securitization to be large.

**Appendix D. Parametric Models with Invariant Divergences**

In this section of the appendix, I discuss a parametric moral hazard problem with an invariant divergence cost function. The moral hazard models of Innes [1990], Hellwig [2009], and Fender and Mitchell [2009] can all be thought of as restriction the seller’s choice of probability distributions to a parametric model. This discussion will also allow me to consider “almost non-parametric” models, in which there is a single dimension of aggregate risk that is not controlled by the seller. I will show the the analysis employed in the main text can be used to derive a lower bound on the utility achieved by an security design, and that this lower bound becomes tight as the moral hazard problem becomes increasingly flexible.

I assume that the set of feasible probability distributions, $M_{\xi}$, is a subset of the probability simplex, smooth parametrized by the parameters $\xi$. Formally, in the terminology of Amari and Nagaoka [2007], $M_{\xi}$ is a smoothly embedded sub-manifold of the simplex, consisting of a (possibly

\textsuperscript{23}After about one year, ~11% of securitized loans were in default, compared to ~8% of loans held in portfolio.
curved) exponential family of probability distributions. This is almost without loss of generality—the key restriction is the smoothness.

I will rewrite the parametric moral hazard sub-problem as

\[
\phi(\eta; M_\xi, \theta^{-1}) = \max_{p \in M_\xi} \left\{ \sum_{i > 0} p^i \eta_i - \psi(p; \theta^{-1}) \right\}.
\]

I assume the seller controls at least one parameter that alters the mean asset value, so the problem is not trivial. As noted earlier, in the parametric model there is no guarantee that there is a unique \( p \) which maximizes equation (D.1). I assume for all retained tranches \( \eta_i \), for each optimal \( p \), the parameters \( \xi \) corresponding to that \( p \) are interior. As in the previous section, I will assume that

\[
\psi(p) = \theta c^{-1} D(p||q)
\]

where \( D(p||q) \) is an invariant divergence and \( c \) is defined as in Chentsov’s theorem, and continue to assume that \( D \) is smooth and convex in \( p \). I also assume that \( q \in M_\xi \), meaning that the “zero-cost” distribution is feasible. Let \( U(s; M_\xi, \theta^{-1}) \) be the utility in the security design problem,

\[
U(s; M_\xi, \theta^{-1}, \kappa) = \beta s(1 + \kappa) \sum_{i > 0} p^i(\eta(s); M_\xi, \theta^{-1}) s_i + \phi(\eta(s); M_\xi, \theta^{-1}),
\]

where \( p(\eta; M_\xi, \theta^{-1}) \) is the endogenous probability the seller would choose given \( \eta, M_\xi, \) and \( \theta^{-1} \). I have written \( p(\eta; M_\xi, \theta^{-1}) \) as a function, assuming an arbitrary rule for choosing between different optimal \( p \), in the event that there are multiple maximizers of equation (D.1). Unlike the non-parametric models considered previously, the mapping between securities and probability distributions is many-to-one. Moreover, even under the approximation considered in section §5, the optimal contract does not approach debt. For example, in the Innes [1990] problem, the optimal contract in this limit is still a “live-or-die” contract. Nevertheless, I use the approximation to derive an analog to appendix proposition 11 for parametric models:

**Proposition 9.** In the parametric model, with a smooth, convex, invariant divergence cost function, the difference in utilities achieved by an arbitrary security \( s \) and the sell-nothing security, for sufficiently small \( \theta^{-1} \) and \( \kappa \), is bounded below:

\[
U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) \geq \kappa E^q[\beta s_i] - \frac{1}{2} \theta^{-1} V^q[\beta s_i] + O(\theta^{-2} + \kappa \theta^{-1}; M_\xi).
\]

The lower bound, to first order, does not depend on \( M_\xi \), only on \( q \). There exist parameters \( \xi \) and corresponding set of feasible probability distributions \( M_\xi \) such that the lower bound is tight. The
notation $O(\theta^{-2} + \kappa\theta^{-1}; M_\xi)$ indicates terms of order $\theta^{-2}$, $\kappa\theta^{-1}$, or higher order that may depend on $M_\xi$.

Proof. See appendix, section G.11.

In the parametric model, there is still a mean-variance tradeoff between the gains from trade and the losses due to both reduced effort and risk-shifting. However, because the model is parametric, there are ways that the security can vary that do not alter the incentives of the seller, because there is no action she can take to respond to this variation. The seller’s limited ability to respond to changing incentives explains why the result in proposition 9 is a lower bound.24

This lower bound applies to a large class of problems in the existing security design literature. For problems with a single choice variable, such as Innes [1990], the restriction to invariant divergence cost functions is almost without loss of generality, because all $f$-divergences are invariant, and the associated $f$-function can be any smooth, convex function. The lower bound even applies to problems where the standard “first-order-approach” is not valid (Jewitt [1988], Mirrlees [1999]). Although parametric moral hazard sub-problems do not always have unique or everywhere-differentiable solutions, in the neighborhood of $\theta^{-1} \to 0^+$ the policy function $p(\eta; M_\xi, \theta^{-1})$ becomes unique and differentiable for all smooth manifolds $M_\xi$. The lower bound can therefore be thought of as a tractable approach to approximating problems that are otherwise difficult to analyze.

The lower bound is tight, in the sense that there exists an $M_\xi$ such that the utility difference is equal to the mean-variance objective, up to order $\theta^{-1}$. The example that illustrates this provides an interesting economic intuition. Take the security $s$ and corresponding retained tranche $\eta$ as given. Suppose that $M_\xi$ is an exponential family,

$$p^i(\xi) = q^i \exp(\xi_1 \eta_i + \xi_2 s_i - A(\xi)),$$

where $\eta$ and $s$ are the sufficient statistics of the distribution25, and $A(\xi)$ is the log-partition function that ensures $p^i(\xi)$ is a probability distribution for all $\xi$. The zero-cost distribution, $q^i$, corresponds to $\xi_1 = \xi_2 = 0$. This example is a “worst-case scenario” for the agents, over the set of possible actions, holding the security design fixed. It is the worst case because the exponential family can also be expressed as a function of its dual coordinates, $\tau$, with $\tau_1(\xi) = \sum_i p^i(\xi) \eta_i$ and $\tau_2(\xi) = \sum_i p^i(\xi) s_i$ (Amari and Nagaoka [2007]). In effect, the seller separately controls the value of her

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24 The proof uses the monotonicity property of the Fisher information metric, and is closely related to the Cramér-Rao bound.

25 Informally, the sufficient statistics capture all of the information in a data sample. In this case, if an observer saw the value of both the retained tranche and security, there would be no additional information that would be useful when trying to infer the seller’s actions.
retained tranche and the security. The seller will use only the action that increases the value of her retained tranche ($\tau_1$) and will not use the action that increases the value of the security ($\tau_2$). Relative to the case where the seller retained the entire asset, this situation exhibits both reduced effort and risk-shifting.

The lower bound in this example is tight, and there is no “costless variation,” because the restriction of the seller’s actions to this particular parametric family did not alter the probability distribution $p(\eta)$ that she would choose, relative to the non-parametric model. The proof of proposition 9 relies on the fact that the security $s$ is a sufficient statistic of the distribution, and that sufficient statistics (as estimators) attain the Cramér-Rao bound.

It is interesting to note the contrast between the worst-case scenarios for debt securities and equity securities. For debt securities, the worst-case scenario is that the seller controls an option-like payoff, which increases both the mean and variance of the asset value. For equity securities, the worst-case scenario is that the seller controls the mean outcome.

The security design utility in the parametric problem, under the approximation, exhibits a monotonicity with respect to the set of feasible distributions $M_\xi$.\footnote{See appendix section G.11, which proves this.} If $M_\xi \subseteq \hat{M}_\xi$, then for all $s$, and sufficiently small $\theta^{-1}$ and $\kappa$,

$$U(s; M_\xi, \theta^{-1}, \kappa) \geq U(s; \hat{M}_\xi, \theta^{-1}, \kappa).$$

Earlier, I assumed that the seller could influence the mean of the probability distribution. It follows that giving the seller additional actions can only, under the first-order approximation, expand the scope for risk-shifting, and therefore reduce the utility that can be achieved in the security design problem. Eventually, as the set of feasible probability distributions approaches the entire probability simplex, the problem converges to the non-parametric case, and the lower bound is tight for all securities. In this case, as in the previous section, debt is optimal.

The approximate lower bound in proposition 9 (the mean and variance terms) does not depend on the parameters $\xi$, which suggests an interpretation of debt as a robust security design. The robustness of debt is complementary to the result of Carroll [2015]. The key difference between this model and Carroll [2015] is the cost of each potential probability distribution, $p$. In Carroll [2015], each probability $p$ could have different, arbitrary costs. When there is no structure on the cost of potential actions, risk-shifting concerns dominate concerns about reduced effort, and it is crucial that the buyer and seller’s payoffs be exactly aligned. In this case, the optimal security is equity. By contrast, the invariant divergence cost assumption in this model imposes a structure on the cost that any probability distribution $p$ would have, if that $p$ were feasible. With the cost
structure I have imposed, both reduced effort and risk-shifting are potentially important, the mean-variance intuition applies, and debt is approximately optimal. Together, these theories can explain the prevalence of both debt and equity securities. Debt securities are used when concerns about reduced effort and risk-shifting are both relevant. If concerns about risk-shifting dominate, equity securities are used.

In this discussion, I have emphasized the application of the parametric framework to models in which the seller controls a small number of parameters. However, we can also use the result to consider an “almost non-parametric” model. Assume that each state $i \in \Omega$ contains information about the idiosyncratic outcome $v_i$ and an aggregate state. In the almost non-parametric model, the seller controls the conditional probability distribution of the asset value for each aggregate state, but does not control the probability distribution of the aggregate states.

If the cost function is the KL divergence, the optimal security design will be an aggregate-state contingent debt security (effectively, there is a separate security design problem for each aggregate state). Nevertheless, the lower bound results of proposition 9 apply, and a non-contingent debt security maximizes this lower bound. These results suggest that it is possible to bound the utility losses of using a non-contingent debt security instead of an aggregate-state contingent debt security.

One application of this idea relates to currency choice. If we assume that the cost function is a pecuniary cost, it follows that the non-contingent debt security is a debt in the currency of the cost function. For example, if a mortgage originator creates loans in the United States, and sells the loans to a euro zone bank, the security design maximizing proposition 9 is a dollar-denominated debt, not a euro-denominated (or yen-denominated, ...) debt. The optimal security design likely depends on aggregate outcomes, including exchange rates, but a non-contingent, dollar-denominated debt may achieve nearly the same utility.

In the next section of the appendix, section §E, I provide a micro-foundation for a version of the parametric model, using a model of rational inattention. The rational inattention model motivates my study of parametric models with the KL divergence cost function. The exact solution for the model cannot be characterized analytically, whereas the approximately optimal debt contract is simple to describe.

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27 In the “almost non-parametric” model, the aggregate state changes the conditional distribution of the idiosyncratic outcomes. The aggregate state does not affect the agents’ relative marginal utilities, or the cost to the seller of manipulating the idiosyncratic outcome. However, the model could easily be extended to accommodate these cases, and the result (an aggregate-state-contingent debt contract) would still hold.

28 This interpretation was suggested to me by Roger Myerson.
Appendix E. Contracting with a Rationally Inattentive Seller

In this section, I consider the problem of contracting with a rationally inattentive seller. I will show that this model is very close to the parametric security design model discussed in section §D, and that the lower bound result in proposition 9 applies.

One possible economic motivation for this type of problem comes from mortgage origination. Suppose that there is a set of $A$ alternative groups of borrowers that the seller (a mortgage originator) could lend to. These groups could be borrowers from different zip codes, with different credit scores, etc., and a single borrower might belong to multiple groups. The seller can choose only one of these groups to lend to, and create the asset by making loans to this group. I assume that the seller can freely observe to which groups a potential borrower belongs, but is uncertain about which group she should lend to. I also assume that, under the seller’s and buyer’s common prior, lending to each alternative group will result in an asset with the same expected value and the same variance of asset value.

The buyer observes which group the seller chose to lend to. The buyer can also observe (in the second period) the ex-post returns of each group, not just the group the seller picked. The set of possible states of the world is $X$, where each state $x \in X$ corresponds to a set of asset values for each of the alternative groups, if that group had been chosen by the seller. There are $N = |A| \cdot |X|$ possible combinations of states and choices by the seller. Let $i$ be an index of these state-choice pairs, $i \in \{0, \ldots, N - 1\}$, and let $v_i$ denote the value of the asset the seller actually created in state-choice $i$. I will assume that $v_0 = 0$.

The value of the security that the seller offers to the buyer can depend on $i$, the state-choice pair. In this sense, the security can be “benchmarked,” and the payoff can depend not only on the ex-post value of the asset the seller created, but also on the ex-post value of the other alternatives she might have chosen. However, limited liability still applies, based on the value of the asset the seller actually created. I will denote the value of the security in state-choice $i$ as $s_i$, and require $s_i \in [0, v_i]$.

The moral hazard in this problem comes from the seller’s ability to acquire information about which states $x \in X$ are mostly likely, before choosing amongst the $A$ alternatives. The quantity and nature of the information the seller acquires is not contractible, and this is what creates the moral hazard problem. The seller could choose an information structure to maximize the expected value of the assets she creates, but will instead choose an information structure to maximize the value of the retained tranche $\eta$. I will model the choice and cost of different information structures using the rational inattention framework of Sims [2003] and the results about discrete choice and rational inattention found in Matějka and McKay [2011].
As in previous sections, I will assume that the seller has a lower discount factor than the buyer, creating gains from trade. For expository purposes, I will use the “principal-agent” timing convention. The timing of the model is as follows. During the first period:

1. The seller designs the security $s_i$, subject to limited liability.
2. The seller makes a take-it-or-leave-it offer to the buyer, at price $K$.
3. The buyer accepts or rejects the offer.
4. The seller chooses a signal structure, unobservable to the buyer.
5. The seller receives a signal about the likelihood of each state in $X$.
6. The seller chooses an alternative $a \in A$.

The seller’s choice of information structure and action is similar to a version of the parametric models discussed earlier. Let $g(x)$ denote the common prior over the possible states $x \in X$. Following the standard simplification result used in rational inattention problems, I will think of the seller as choosing the conditional probability distribution of alternatives $a \in A$, for each state $x$. This model is parametric, because the seller cannot create any probability distribution over the $N$ state-choice combinations. Instead, she is constrained to create only those probability distributions over the $N$ state-choices that have the marginal distribution over states $x \in X$ equal to $g(x)$. Economically, the mortgage originator does not control the amount that each group of mortgage borrowers will repay (the state). The best the mortgage originator can do is learn about which states are likely, and choose which group of borrowers to lend to accordingly.

The cost of the signal structure is described by mutual information, which can be rewritten as a KL divergence. The seller’s moral hazard problem is

$$\phi_{RI}(\eta; \theta^{-1}) = \max_{p \in M_{RI}} \left\{ \sum_i p_i \eta_i - \theta D_{KL}(p || q(p)) \right\},$$

where $M_{RI}$ denotes the set of probability distributions over state-choices with the marginal distribution over states equal to $g(x)$, and $q(p)$ is the joint distribution between states and choices, when choices are independent of states and the marginal distribution of choices in $q$ is the same as the marginal distribution in $p$. The constant of proportionality $\theta$ is now interpreted as the cost per unit of information that the seller acquires. I will continue to assume that the solution is interior and unique. This assumption simplifies the proof and discussion, but can be relaxed (Matějka and McKay [2011] provide more primitive conditions under which the assumption would hold).

The key difference between this moral hazard problem and the parametric problem discussed previously is that the $q$ distribution is endogenous, not fixed. This endogeneity is a property of the mutual information cost function. If the seller were to choose $p$ so that her action were independent of the state, there would be no information cost, regardless of the marginal distribution of actions.
Economically, if the mortgage originator decided to randomly choose a group of borrowers to give loans to, independent of economic fundamentals, this would require no costly information gathering, regardless of how the mortgage originator decided to randomize.

The security design problem is almost identical to the parametric moral hazard model. For an arbitrary security \( s \),

\[
U_{RI}(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) \sum_{i>0} p^i(\eta(s); \theta^{-1}) s_i + \phi_{RI}(\eta(s); \theta^{-1}),
\]

where \( p(\eta; \theta^{-1}) \) is the probability distribution over state-choices that the seller will choose, given retained tranche \( \eta \) and cost of information \( \theta \). Despite the aforementioned complication that \( q \) is endogenous, the lower bound result of proposition 9 applies.

**Proposition 10.** In the rational inattention model, the difference in utilities achieved by an arbitrary security \( s \) and the sell-nothing security, for sufficiently small \( \theta^{-1} \) and \( \kappa \), is bounded below:

\[
U_{RI}(s; \theta^{-1}, \kappa) - U_{RI}(0; \theta^{-1}, \kappa) \geq \kappa \bar{p}[\beta_s s_i] - \frac{1}{2} \theta^{-1} V[p[\beta_s s_i]] + O(\theta^{-2} + \kappa \theta^{-1}),
\]

where \( \bar{p} \) is the endogenous probability distribution that the seller would choose, if they could not gather information and the security was \( s \).

**Proof.** See appendix, section G.12. \( \square \)

The statement of proposition 10 discusses the endogenous probability distribution \( \bar{p} \), which is the probability distribution the seller would choose if they could only choose the unconditional distribution of actions, and the actions were independent of the state. The key intuition is that different security designs can change the seller’s incentives to pick one alternative over another, even if the seller could not gather any information at all. This could be true because the security treats different alternatives differently, or because the security’s valuation depends on third or higher moments of the value distribution (only the mean and variance of each alternative under the prior are assumed to be identical). Debt maximizes the lower bound in proposition 10 only when small, symmetric perturbations to the debt security do not alter the unconditional distribution of actions.\(^{29}\)

This would hold true if the prior distribution of values were identical for each alternative.

In general, rational inattention problems will not result in debt as the optimal contract, because the seller does not control the state \( x \in X \). If there are some states that offer higher payoffs than others, regardless of the alternative chosen, it is not efficient for the seller to receive higher cash flows in those “good” states. Additionally, the security design influences the unconditional

\(^{29}\)By symmetric perturbations, I mean those perturbations that, like the debt contract, depend only on the value of the outcome, and not the alternative chosen.
probability that a particular alternative will be chosen. A debt contract, whose payoff does not vary by alternative chosen, only by outcome, may result in a sub-optimal choice of unconditional probabilities for different actions.

The lower bound is tight for symmetric contracts under two conditions.\(^{30}\) The first is the exchangeable prior discussed by Matějka and McKay [2011]. When the prior is exchangeable, there is no reason for the security payoff to vary by alternative chosen, because each alternative is ex-ante equivalent. The second condition is that the states are symmetric, and there is no such thing as a “good state” or “bad state,” only good and bad alternatives conditional on the state. One justification for this second condition is complete markets (see the discrete-outcome, complete asset market described in He [1990]). Under these two conditions, the lower bound is tight, and a debt security is approximately optimal.\(^{31}\)

Contracting problems with rationally inattentive agents are challenging to solve exactly, both analytically and computationally. The lower bound result of proposition 10 shows that, when the seller’s ability to gather information is weak, debt contracts are a tractable, detail-free way of guaranteeing a minimum utility level.

**APPENDIX F. FREE DISPOSAL AND FREE RISK-SHIFTING**

In this section, I will discuss the impact that free disposal of output by the seller and free risk-shifting would have on the models discussed in the main. For the static, non-parametric moral hazard problems discussed in sections 3, 4, and 5, the optimal security designs feature monotone retained tranches. In the proofs, in lemma 3, I show this is true for any static, non-parametric security design problem with an invariant divergence cost function whose gradient (in \(p\)) is continuous in \(q\).

Intuitively, because the optimal retained tranche is monotone even without free disposal, allowing for free disposal does not change the optimal security design. To see this formally, I will show that, with free disposal, it is without loss of generality to consider monotone retained tranches and ignore the disposal option. Imagine that there is free disposal. We can write the agent’s moral hazard problem as

\[
\phi(\eta) = \sup_{p \in F(r), r \in M} \left\{ \sum_{i > 0} \eta_i p^i - \psi(r) \right\};
\]

where \(F(r)\) is the set of probability distributions first-order stochastically dominated by \(r\), under the ordering given by \(\Omega\). The agent, in effect, makes two choices-- first choosing \(r\) using the technology discussed in the text, then following a (possibly random) output destruction strategy to

\(^{30}\)These conditions are sufficient, not necessary.

\(^{31}\)In fact, under these two conditions, a debt security is optimal, not just approximately optimal.
create $p$. The buyer still receives payoff $\beta b E^p[s]$, and therefore the security design utility described in equation (2.2) is still valid.

Define, for any retained tranche $\eta$, the “monotone version”

$$\bar{\eta}_i(\eta) = \max_{j \in \{0, \ldots, i\}} \eta_j.$$  

Note that, because $v_i$ is weakly increasing in $i$, such a design does not violate the limited liability constraints. Note also that, because of the monotonicity of $\bar{\eta}_i(\eta)$,

$$\sum_{i > 0} [p^i(\eta) - r^i(\eta)] \bar{\eta}_i(\eta) = 0.$$  

We can rewrite the moral hazard problem as

$$\phi(\eta) = \sup_{p \in F(r), r \in M} \left\{ \sum_{i > 0} (\eta_i - \bar{\eta}_i(\eta)) p^i + \sum_{i > 0} \bar{\eta}_i(\eta) r^i - \psi(r) \right\}.$$  

It immediately follows that the behavior without output destruction is the same for the two securities: $r(\eta) = r(\bar{\eta}(\eta))$.

By the definition of the retained tranche, if $\eta_i < \eta_j$ for some $i > j$, then $s_i > s_j$. As a result, output destruction hurts the value of the buyer’s security:

$$\sum_{i > 0} [p^i(\eta) - r^i(\eta)] s_i(\eta) \leq 0$$  

for all $\eta$. Therefore, utility in in the security design problem is weakly higher under $\bar{\eta}(\eta)$ than under $\eta$, and it is without loss of generality to consider monotone security designs.

I have shown that free disposal does not affect the static problems discussed in the text– it is equivalent to a restriction to monotone security designs in the absence of free disposal, and the optimal security designs were monotone even without such a restriction. Conveniently, essentially the same proof applies to the dynamic security design problems.

Suppose we modify the stochastic process for the asset value described in section §6 to allow for output destruction:

$$dV_t = b(V_t, t) dt + u_t \sigma(V_t, t) dt - dY_t + \sigma(V_t, t) dW_t,$$

where $dY_t \geq 0$ is the seller’s destruction of asset value at time $t$. To allow such a modification, we need use as the space of asset values processes the space of RCLL functions on $[0, 1]$, which I will denote $\bar{\Omega}$, instead of the space of continuous functions, which I will continue to denote $\Omega$. We also need to allow the security design to be a function on $\bar{\Omega}$.

I will say that a retained tranche is monotonic in asset value if, for all $t \in [0, 1]$, and all $V \in \bar{\Omega}$, $\eta(V)$ is weakly increasing in $V_t$. Using this definition, debt contracts are monotonic in asset value.
It follows immediately that if the seller is given a retained tranche that is monotonic in asset value, she will not destroy asset value. We can define the “monotone version” of the retained tranche in the following way. Let $F(V)$ be the set of all RCLL functions on $[0, 1]$ for which, for all $f \in F(V)$ and $t \in [0, 1]$, 

$$f_t \leq V_t.$$ 

The monotone version of $\eta(V)$ is 

$$\bar{\eta}(\eta, V) = \sup_{f \in F(V)} \eta(f).$$ 

Note that, because $f_1 \leq V_1$, this retained tranche satisfies the limited liability constraints.

The “weak formulation” approach, based on Girsanov’s theorem and described in proposition 5, can be applied. We can defined an alternative probability space, with measure $Q$, on which 

$$dX_t = b(X_t, t)dt - dY_t + \sigma(X_t, t)dB_t,$$

and a measure $P$, absolutely continuous with respect to $Q$, such that, under $P$, $X$ has the same law as $V$ under measure $\tilde{P}$.

Suppose that the retained tranche is not monotonic in asset value. There is some $t$ and some $X$ such that, if the seller reaches state $(t, X_t)$, she will wish to destroy output. If such a state is never reached with positive probability under measure $P(\eta)$ (and hence $Q$), the retained tranche and its monotone version achieve the same utility in the security design problem, holding the measure $P(\eta)$ constant. Such a state can never be reached under any measure that is absolutely continuous with respect to $Q$, and therefore the monotone version of the retained tranche will not affect the agent’s choice of $P$. It follows, in this case, that it is without loss of generality to assume monotonicity.

Assume, going forward, that if a non-monotonicity exists, it is reached with positive probability. I will show that, for any retained tranche that induces the seller to destroy some asset value, there is another retained tranche that does not induce the seller to destroy asset value and achieves higher utility in the security design problem. As a result, the optimal security design is monotone.

Define a modified version of the retained tranche in the following way: for each $B \in \Omega$, let $X^Y(\eta, B) \in \Omega$ denote the asset value path that occurs under the seller’s optimal output destruction plan, given retained tranche $\eta$ and brownian motion $B$, and let $X(B)$ be the asset value path that would occur in the absence of output destruction. Note that $X(B)$ is not affected by the design of the retained tranche, and that there is a one-to-one mapping between $X$ and $B$. We can defined a modified version of the retained tranche, for $X \in \Omega$, as 

$$\tilde{\eta}(X, \eta) = \eta(X^Y(\eta, B(X))).$$
where $B(X)$ is the Brownian motion that induces $X$ in the absence of asset value destruction. For discontinuous $X$, let $\tilde{\eta}(X, \eta) = 0$. Note that, because asset value destruction decreases $X_1$, this modified retained tranche satisfies the limited liability constraints.

By revealed preference, $\tilde{\eta}(X)$ does not induce output destruction— if it did, the seller’s output destruction given $\eta$ would not have been optimal. Moreover, $\tilde{\eta}$ must also induce the same choice of $P$; again, if some different choice of $P$ was preferable, it would also be preferable under the contract $\eta$. It follows that the seller receives the same utility from $\eta$ and $\tilde{\eta}$.

The buyer, however, receives weakly higher utility from $\tilde{\eta}$. By the assumptions discussed in section §6, for any realization of the Brownian motion $\omega \in \Omega$, destruction of output at time $t$ lowers the value of the asset for all times $s \geq t$, relative to the asset values that would have been generated in the absence of destruction. $E_t[V_1]$ is always decreased by destruction (by assumption), and $s(X) = X_1 - \beta_s^{-1}\eta(X)$. As a result, $\tilde{\eta}$ delivers weakly higher utility than $\eta$, and it is without loss of generality to study monotone security designs and assume no output destruction.

Finally, I will discuss “free risk-shifting.” As discussed in the text, the strict convexity assumption on the divergences I study rules out risk-shifting that is completely free. One implication of free risk-shifting is that there is not necessarily a unique optimal probability distribution for the seller to choose in the moral hazard problem. For the purposes of discussion, suppose that there is some convention by which a single $p(\eta)$ is determined for each $\eta$. This convention can also be used to choose a unique $p_e(e)$ for each level of effort.

The decomposition described in lemma 1 applies with free risk-shifting, except that the divergence terms cancel. It can be rewritten in a slightly more useful form:

$$U(\eta) = \beta_b \sum_{i \in \Omega} q_i v_i - \kappa \sum_{i \in \Omega} p^i(\eta)[\eta_i - \gamma(\eta)\beta_s v_i] + \frac{\beta_b}{\beta_s} e(\eta) - c(e(\eta)) - \kappa \sum_{i \in \Omega} p^i_e(e(\eta))\gamma(\eta)\beta_s v_i.$$

The moral hazard problem can be written as

$$\phi(\eta) = \max_{e, p \in M(e)} \sum_{i \in \Omega} p^i(\eta)[\eta_i - \gamma(\eta)\beta_s v_i] + \sum_{i \in \Omega} p^i_e(e(\eta))\gamma(\eta)\beta_s v_i - c(e),$$

where $M(e) \subset M$ is the set of probability distributions associated with effort level $e$.

By the seller’s optimal choice of $p$ in the moral hazard problem, it must be the case that

$$\sum_{i \in \Omega} p^i(\eta)[\eta_i - \gamma(\eta)\beta_s v_i] \geq 0.$$
It follows immediately that the equivalent equity tranche delivers higher utility in the security design problem, if it is feasible. By limited liability, \( 0 \leq \eta_i \leq \beta_s v_i \), and therefore the effort level implemented is less than the level of effort when selling nothing, and more than the effort level implemented by selling everything, and hence is feasible. It follows that, with free risk-shifting, an equity security is always an optimal security design.

**APPENDIX G. PROOFS**

G.1. **Proof of lemma 1.** The utility of an arbitrary security design is

\[
U(\eta) = \beta_b \sum_{i \in \Omega} p^j(\eta) s_i(\eta) + \phi(\eta)
\]

\[
= \beta_b \sum_{i \in \Omega} p^j(\eta) s_i(\eta) + \sum_{i \in \Omega} p^j(\eta) \eta_i - D(p(\eta) \| q).
\]

The security and retained tranche add up to the asset value:

\[
\beta_b v_i = \beta_b s_i + \frac{\beta_b}{\beta_s} \eta_i.
\]

Therefore,

\[
U(\eta) = \beta_b \sum_{i \in \Omega} p^j(\eta) v_i - \left( \frac{\beta_b}{\beta_s} - 1 \right) \sum_{i \in \Omega} p^j(\eta) \eta_i - D(p(\eta) \| q).
\]

Using the definition of \( \kappa \), and adding and subtracting terms,

\[
U(\eta) = \beta_b \sum_{i \in \Omega} q^j v_i + \frac{\beta_b}{\beta_s} \sum_{i \in \Omega} (p^j(\eta) - q^j) v_i - D(p_s(e(\eta)) \| q) - \kappa \sum_{i \in \Omega} p^j_s(e(\eta)) \eta_i +
\]

\[
D(p_s(e(\eta)) \| q) - D(p(\eta) \| q) - \kappa \sum_{i \in \Omega} (p^j(\eta) - p^j_s(e(\eta))) \eta_i.
\]

By definition,

\[
\beta_s \sum_{i \in \Omega} (p^j(\eta) - q^j) v_i = e
\]

and

\[
D(p_s(e(\eta)) \| q) = c(e),
\]

which proves the utility decomposition result.

Next, I will prove that the risk-shifting component is always strictly negative if \( \eta \) is not equal to \( \gamma(\eta) \beta_s v \). Note that it is zero, by construction, if that equality holds. By optimality in the seller’s
moral hazard problem,
\[ \sum_{i \in \Omega} p^i(\eta) \eta_i - D(p(\eta)||q) > \sum_{i \in \Omega} p^i_e(e(\eta)) \eta_i - D(p_e(e(\eta))||q), \]
where the inequality is strict by the strict convexity of \( D \) and the assumption that \( \eta \neq \gamma(\eta) \beta_s v \) (strictly convex functions have unique maxima over convex sets). It follows that
\[ D(p_e(e(\eta))||q) - D(p(\eta)||q) - \kappa \sum_{i \in \Omega} (p^i(\eta) - p^i_e(e(\eta))) \eta_i < (1 + \kappa)[D(p_e(e(\eta))||q) - D(p(\eta)||q)]. \]

By definition, \( D(p_e(e(\eta))||q) \) is minimal over the set of \( D(p||q) \) associated with effort level \( e(\eta) \), which includes \( p(\eta) \), and therefore
\[ D(p_e(e(\eta))||q) - D(p(\eta)||q) - \kappa \sum_{i \in \Omega} (p^i(\eta) - p^i_e(e(\eta))) \eta_i < 0. \]

Now consider a perturbation to the security design problem. Suppose that the retained tranche was the equivalent equity security, both before and after the perturbation. In that case, we would have
\[ U(\gamma, e(\gamma)) = \beta_b \sum_i q^i v_i + \frac{\beta_b}{\beta_s} e - c(e) - \kappa \gamma \beta_s \sum_i p^i_e(e) v_i. \]

The indirect effect would be
\[ \beta_b \sum_{j \in \Omega} s^j \left. \frac{dp^j(\eta(\epsilon))}{d \epsilon} \right|_{\epsilon=0^+} = \frac{\beta_b}{\beta_s} \sum_{j \in \Omega} (\beta_s v_j - \gamma(\eta^*) \beta_s v_j) \left. \frac{dp^j_e(e(\epsilon))}{d \epsilon} \right|_{\epsilon=0^+} = \frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \left. \frac{d e}{d \epsilon} \right|_{\epsilon=0^+}, \]
where the latter follows by the definition of effort. Note that this is also the derivative of \( U(\gamma, e(\gamma)) \) with respect to effort, holding \( \gamma \) constant.

Now consider a more general perturbation. We can decompose the indirect effect as
\[ \beta_b \sum_{j \in \Omega} s^j \left. \frac{dp^j(\eta(\epsilon))}{d \epsilon} \right|_{\epsilon=0^+} = \frac{\beta_b}{\beta_s} \sum_{j \in \Omega} (\beta_s v_j - \eta^*_j) \left. \frac{dp^j_e(\eta(\epsilon))}{d \epsilon} \right|_{\epsilon=0^+}. \]

By the definition of effort,
\[ \sum_{j \in \Omega} \beta_s v_j \left. \frac{dp^j(\eta(\epsilon))}{d \epsilon} \right|_{\epsilon=0^+} = \left. \frac{d e(\eta(\epsilon))}{d \epsilon} \right|_{\epsilon=0^+}. \]
It follows that
\[ \beta_b \sum_{j \in \Omega} s_j^* \frac{dp_j^*(\eta^*)}{d\epsilon} \bigg|_{\epsilon=0^+} = \frac{\beta_b}{\beta_s} \frac{de(\eta^*)}{d\epsilon} - \frac{\beta_b}{\beta_s} \sum_{j \in \Omega} \eta_j dp_j(\eta^*) \bigg|_{\epsilon=0^+}, \]
and that
\[ \beta_b \sum_{j \in \Omega} s_j^* \frac{dp_j^*(\eta^*)}{d\epsilon} \bigg|_{\epsilon=0^+} = \frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{de(\eta^*)}{d\epsilon} - \frac{\beta_b}{\beta_s} \sum_{j \in \Omega} (\eta_j^* - \gamma(\eta^*)\beta_s v_j) dp_j(\eta^*) \bigg|_{\epsilon=0^+}. \]

G.2. Proof of proposition 2. For any f-divergence, the Hessian is
\[ \frac{\partial^2 \psi(p)}{\partial p_i^2} \bigg|_{p=p(\eta^*)} = \theta \left( \frac{\delta_{ij}}{q^i} f''(\frac{p_i^*(\eta^*)}{q^i}) + \frac{1}{q^0} f''(\frac{p_0^*(\eta^*)}{q^0}) \right). \]
To prove proposition 2, I suppose that there is some f-divergence such that, for every sample space \( \Omega \) and zero-cost distribution \( q \), the optimal security design is debt. I will show that this f-divergence must be the KL divergence.

Take some \( \Omega \) and \( q \) such that the optimal security design is a debt, with at least two distinct indices \( j \) and \( j' \) such that \( s_j = s_j' \) and \( 0 < s_j < \min(v_j, v_j') \), and such that the solution to the moral hazard problem under the optimal contract is interior. The indices \( j \) and \( j' \) correspond to the “flat” part of the debt contract. I construct an \( \Omega \) and \( q \) with this property below, showing that such examples exist. Because the solution to the moral hazard problem is interior, from the security design equation,
\[ s_j(\eta^*) = \sum_{i>0} \theta(\beta_s^{-1} - \beta_b^{-1})[p_i^*(\eta^*) - \lambda^i + \omega^i]\left[ \frac{\delta_{ij}}{q^i} f''(\frac{p_i^*(\eta^*)}{q^i}) + \frac{1}{q^0} f''(\frac{p_0^*(\eta^*)}{q^0}) \right]. \]
For the indices \( j \) and \( j' \) associated with the flat part of the debt contract, neither multiplier binds, and therefore
\[ \frac{p_i^*(\eta^*)}{q^i} f''(\frac{p_i^*(\eta^*)}{q^i}) = \frac{p_j^*(\eta^*)}{q^j} f''(\frac{p_j^*(\eta^*)}{q^j}). \]
If the optimal security is a debt for some \( \Omega \) and \( q \), this property must hold for all \( j \) and \( j' \) associated with the flat part of that debt contract. Below, I state a claim. I will first prove the rest of the proposition, assuming this claim, and then prove the claim.

Claim 1. For all pairs \( u_1 \in (0, 1] \) and \( u_2 \in [1, \infty) \), there exist \( \Omega \) and \( q \) such that, under the optimal debt contract, there are indices \( j \) and \( j' \), associated with the flat part of the debt contract, such that
\[ \frac{p_j^*(\eta^*)}{q^j} = u_1 \]
and
\[ \frac{p_j^i(\eta^\ast)}{q^i} = u_2. \]

Assume the above claim is true. If the optimal security is a debt, then the optimal debt contract (the best contract in the class of debt contracts) is the optimal security (in the space of all securities). Because debt is optimal for all \( \Omega \) and \( q \), including the ones constructed in the above claim, it follows that for any \( u_1 \) and \( u_2 \), we must have

\[ u_1 f''(u_1) = u_2 f''(u_2). \]

When \( u_1 = 1 \), \( u f''(u) = 1 \) by the normalization assumption, and therefore for all \( u \in (0, \infty) \),

\[ u f''(u) = 1. \]

Solving the differential equation, using the normalization assumptions that \( f'(1) = 0 \) and \( f(1) = 0 \),

\[ f(u) = u \ln u - u + 1 \]

for all \( u \in (0, \infty) \).

Now consider \( f(0) \). By convention, for the KL divergence, \( f(0) = 0 \). By assumption, \( f \) is continuous, and therefore

\[ f(0) = \lim_{\epsilon \to 0^+} f(\epsilon) = 0, \]

proving the proposition. To complete the proof, I prove the above claim.

**G.2.1. Proof of claim 1.** First, note the first-order condition for the moral hazard problem, assuming an interior solution, requires that

\[ f' \left( \frac{p_j^i(\eta^\ast)}{q^i} \right) - f' \left( \frac{p_0^0(\eta^\ast)}{q^0} \right) = \theta^{-1} \eta_i = \theta^{-1} \beta_s (v_i - s_i). \]

For the flat region for some \( j \),

\[ f' \left( \frac{p_j^i(\eta^\ast)}{q^j} \right) - f' \left( \frac{p_0^0(\eta^\ast)}{q^0} \right) = \theta^{-1} \beta_s (v_j - \bar{v}). \]

Let \( \Omega = \{0, v_1, v_2\} \). I will construct, under the optimal debt security,

\[ \frac{p^1(\eta^\ast)}{q^1} = u_1, \]

\[ \frac{p^2(\eta^\ast)}{q^2} = u_2, \]
for given \( u_1 \in (0, 1) \) and \( u_2 \in [1, \infty) \). Assume for now that the optimal debt contract has \( \bar{v} < v_1 \) as its maximum value. By the first-order conditions, we have (using the normalization condition)

\[
f'(u_1) - f'(u_0) = \theta^{-1} \beta_s (v_1 - \bar{v}),
\]

where \( u_0 = \frac{v^0(\eta^*)}{q^0} \). It follows that

\[
f'(u_2) - f'(u_1) = \theta^{-1} \beta_s (v_2 - v_1).
\]

Because \( f(u) \) is convex, this equation can be solved to pin down \( v_2 \) given \( u_2 \) and \( u_1 \), noting that \( v_2 \geq v_1 \) by the convexity and normalization of \( f \), and \( u_2 \geq u_1 \). Next, I will use the adding up constraints to choose a \( q \). We must have

\[
q^0 + q^1 + q^2 = 1
\]

and

\[
q^0 u_0 + q^1 u_1 + q^2 u_2 = 1.
\]

Putting these together,

\[
u_0(q^1, q^2) = \frac{1 - q^1 u_1 - q^2 u_2}{1 - q^1 - q^2}.
\]

The solution will be interior if \( u_0(q^1, q^2) > 0 \). Finally, I define the optimal debt security. Denote the class of retained tranches associated with debt securities as \( \eta(\bar{v}) \). The security design FOC with respect to \( \bar{v} \) is

\[
(1 - \beta_b \beta_s) \sum_{i>0} p^i(\eta^*) \frac{\partial \eta_i(\bar{v})}{\partial \bar{v}} + \beta_b \sum_{i,j>0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j}|_{\eta=\eta^*} s_j(\eta^*) \frac{\partial \eta_k(\bar{v})}{\partial \bar{v}} = 0.
\]

Define \( \bar{v}_{\text{full}}(q^1, q^2) \) as the solution to this equation, given \( q^1 \) and \( q^2 \). The Hessian of \( \phi \) is the inverse of the Hessian of \( \psi \). Using the Sherman-Morrison formula,

\[
\frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j}|_{\eta=\eta^*} = \theta^{-1} \frac{\delta_{ij} q^i q^j}{f''(u_i)} - \theta^{-1} \frac{f''(u_i) f''(u_j)}{\sum_i f''(u_i)}.
\]

Assuming that \( v_1 > \bar{v} \), this simplifies to

\[
(\beta_b^{-1} - \beta_s^{-1})(1 - u_0 q^0) + \beta_b \bar{v} \sum_{i,j>0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j}|_{\eta=\eta^*} = 0.
\]
Define \( \bar{v}(q_1, q_2) \) as the solution to this equation, which differs from \( \bar{v}_{full}(q_1, q_2) \) due to the assumption that \( \bar{v} < v_1 \) and associated simplifications. Summing,

\[
\sum_{i,j > 0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j} \bigg|_{\eta = \eta^*} = \theta^{-1} \sum_{i > 0} \frac{q_i q_j f''(u_i)}{f''(u_{i})} - \theta^{-1} \sum_{i,j > 0} \frac{q_i q_j f''(u_i)}{f''(u_{i})}.
\]

Because \( \phi \) is convex, we can think of this as

\[
\sum_{i,j > 0} \frac{\partial^2 \phi(\eta)}{\partial \eta_i \partial \eta_j} \bigg|_{\eta = \eta^*} = h(q_1, q_2),
\]

where \( h(q_1, q_2) \) is a positive-valued function. Therefore,

\[
\bar{v}(q_1, q_2) = (\beta_s^{-1} - \beta_b^{-1}) \theta q_1 u_1 + q_2 u_2 h(q_1, q_2).
\]

We can define \( v_1 \) from

\[
v_1 = \bar{v}(q_1, q_2) + \theta \beta_s^{-1} (f'(u_1) - f'(u_0(q_1, q_2))),
\]

which is greater than \( \bar{v} \) by the convexity and normalization of \( f \), as long as \( u_0 < u_1 \), which can be maintained by choice of \( q_1 \) and \( q_2 \). It remains to be shown that \( \bar{v}(q_1, q_2) \) is the globally optimal contract, considering also contracts with \( \bar{v} > v_1 \). Note that in the limit as \( q_1 \to 0 \),

\[
\lim_{q_1 \to 0^+} \bar{v}_{full}(q_1, q_2) = \lim_{q_1 \to 0^+} \bar{v}(q_1, q_2),
\]

because whether \( v_1 > \bar{v} \) or not does not alter the utility from the security design, since \( q_1 \) is small. Moreover, in this limit, \( u_0(q_1, q_2) \) can be made arbitrarily small through a choice of \( q_2 \), preserving the requirement that \( u_0(q_1, q_2) \in (0, u_1) \). It follows that there exist \( q_1 \) and \( q_2 \) such that \( u_0(q_1, q_2) < u_1, 0 < \bar{v}(q_1, q_2) < v_1 < v_2 \), and the desired result

\[
\frac{p_1(\eta^*)}{q_1} = u_1,
\]

\[
\frac{p_2(\eta^*)}{q_2} = u_2,
\]

holds.

G.3. **Proof of corollary 2.** The statement of the corollary assume that \( u f''(u) \) is weakly decreasing in \( u \). By our normalization, it follows that

\[
u f''(u) \geq 1
\]
for all \( u \in (0, 1] \). Therefore,
\[
\int_\epsilon^1 f''(u)du \geq \int_\epsilon^1 u^{-1}du,
\]
and by our normalization that \( f'(1) = 0 \),
\[
-f'(\epsilon) \geq -\ln(\epsilon).
\]
It follows that
\[
\lim_{\epsilon \to 0^+} f'(\epsilon) = -\infty.
\]
Suppose there are two points, \( \eta_i \) and \( \eta_j \), with \( v_i \geq v_j \), and such that \( \eta_j \leq \eta_i \). In the moral hazard problem, consider a perturbation that increases the probability of \( p^i \) and decreasing the probability of \( p^j \), holding all other probabilities fixed, and the reverse perturbation. If the solutions for \( p^i \) and \( p^j \) are interior,
\[
\eta_i - \eta_j - \theta f'(\frac{p^i(\eta)}{q^i}) + \theta f'(\frac{p^j(\eta)}{q^j}) = 0.
\]
The existence of interior solutions is guaranteed by the infinite marginal cost shown above. By the assumption that \( \eta_i > \eta_j \), it follows that
\[
\theta f'(\frac{p^i(\eta)}{q^i}) \geq \theta f'(\frac{p^j(\eta)}{q^j}).
\]
By the convexity of \( f \),
\[
\frac{p^i(\eta)}{q^i} \geq \frac{p^j(\eta)}{q^j}.
\]
Now suppose, contrary to the corollary, that the optimal security design is neither nothing, everything, nor a debt contract. It follows by the definition of debt that there exists an \( s_i \) and \( s_j \), with \( i \neq j \), such that \( s_j < v_j \) and \( s_j < s_i \). By weak monotonicity, this implies that \( v_i > v_j \), and therefore (by lemma 3) that \( \eta_i \geq \eta_j \). By the logic above,
\[
\frac{p^i(\eta)}{q^i} \geq \frac{p^j(\eta)}{q^j}.
\]
I will now construct a perturbation that improves welfare. Suppose that we increase \( \eta_i \) by \( \frac{\epsilon}{q^i} f''(\frac{p^i(\eta)}{q^i}) \), and decrease \( \eta_j \) by \( \frac{\epsilon}{q^j} f''(\frac{p^j(\eta)}{q^j}) \). By the assumption that \( s_j < v_j, \eta_j > 0 \), and by assumption, \( s_i > s_j \). It follows that this perturbation is feasible: \( s_i > s_j \), so it won’t violate the buyer’s monotonicity constraint (for sufficiently small \( \epsilon \)), and \( s_i > 0 \) and \( \eta_j > 0 \), so it does not violate limited liability constraints.
For any \( k \notin \{i, j\} \), by the existence of interior solutions,
\[
\eta_i - \eta_k - \theta f'(\frac{p^i(\eta)}{q_i}) + \theta f'(\frac{p^k(\eta)}{q_k}) = 0,
\]
\[
\eta_j - \eta_k - \theta f'(\frac{p^j(\eta)}{q_j}) + \theta f'(\frac{p^k(\eta)}{q_k}) = 0.
\]
Conjecture that
\[
\frac{\partial p^i(\eta(\epsilon))}{\partial \epsilon} = -\frac{\partial p^j(\eta(\epsilon))}{\partial \epsilon} = \theta^{-1},
\]
\[
\frac{\partial p^k(\eta(\epsilon))}{\partial \epsilon} = 0,
\]
and observe that the first-order conditions are satisfied. The welfare implications of this perturbation are (equation (2.3))
\[
\frac{\partial U(\eta(\epsilon))}{\partial \epsilon}\big|_{\epsilon=0^+} = -\kappa \sum_{i \in \Omega} p^i(\eta^*) \frac{\partial \eta_i}{\partial \epsilon}\big|_{\epsilon=0^+} + \beta \sum_{i,j \in \Omega} s_j^i \frac{\partial p^j(\eta)}{\partial \eta_i}\big|_{\eta=\eta^*} \frac{\partial \eta_i}{\partial \epsilon}\big|_{\epsilon=0^+}
\]
\[
= -\kappa (\frac{p^i(\eta^*)}{q_i} f''(\frac{p^i(\eta^*)}{q_i}) - \frac{p^j(\eta^*)}{q_j} f''(\frac{p^j(\eta^*)}{q_j})) + \theta^{-1}(s_i - s_j).
\]
By the weak monotonicity, \( \eta_i \geq \eta_j \), and therefore \( \frac{p^i(\eta^*)}{q_i} \geq \frac{p^j(\eta^*)}{q_j} \). By the assumption that \( uf''(u) \) is weakly decreasing,
\[
\frac{p^i(\eta^*)}{q_i} f''(\frac{p^i(\eta^*)}{q_i}) \leq \frac{p^j(\eta^*)}{q_j} f''(\frac{p^j(\eta^*)}{q_j}).
\]
Combined with the assumption that \( s_i > s_j \), it follows that
\[
\frac{\partial U(\eta(\epsilon))}{\partial \epsilon}\big|_{\epsilon=0^+} > 0.
\]
Therefore, there exists a feasible, utility-improving perturbation, if the security design is not debt, selling everything, or selling nothing.

G.4. Additional Lemmas.

Lemma 3. In the non-parametric security design problem with an invariant divergence cost function, if the gradient of the divergence (with respect to \( p \)) is continuous in \( q \), it is without loss of generality to consider security designs featuring a retained tranche that is monotone in the asset value: \( v_j \geq v_i \) implies that \( \eta_j \geq \eta_i \).

Proof. This proof relies on notions of invariance, which are discussed minimally in the text. Please refer to Chentsov [1982] and Amari and Nagaoka [2007] the definition and properties of invariant functions and divergences.
I will first show that the utility in the security design problem, for an arbitrary security, is continuous in the zero-cost probability distribution, \( q \). The utility can be written as

\[
U(\eta, q) = \beta \sum_{i>0} p^i(\eta, q)s_i(\eta) + \sum_{i>0} p^i(\eta, q)\eta_i - D(p(\eta, q)||q).
\]

It follows that, if \( p(\eta, q) \) is continuous in \( q \), the utility will be continuous in \( q \). The seller’s moral hazard problem is

\[
\max_{p \in \mathcal{M}} \sum_{i>0} p^i \eta_i - D(p||q).
\]

By the convexity of \( D \) (in \( p \)), the KKT conditions are necessary and sufficient, and the FOC can be written as

\[
\eta_i - \partial_i D(p(\eta, q)||q) - \xi + \nu_i = 0,
\]

where \( \xi \) and \( \nu_i \) are the multipliers on the constraints that \( \sum_{i \in \Omega} p^i = 1 \) and \( p^i \geq 0 \), respectively.

By the complementary slackness condition (\( p^i > 0 \) implies \( \nu_i = 0 \)),

\[
\xi = \sum_{i \in \Omega} p^i(\eta, q)(\eta_i - \partial_i D(p(\eta, q)||q)).
\]

Therefore,

\[
\text{(G.1)} \quad \eta_i - \partial_i D(p(\eta, q)||q) - \sum_{i \in \Omega} p^i(\eta, q)(\eta_i - \partial_i D(p(\eta, q)||q)) \leq 0,
\]

with equality if \( p^i(\eta, q) > 0 \). Continuity follows from the continuity of \( \partial_i D(p(\eta, q)||q) \).

Suppose now that \( q \) can be expressed as a vector of rational numbers: for all \( i \in \Omega \), \( q^i = \frac{a^i}{b} \), for some integers \( a^i \) and \( b^i \). Define \( B = \prod_{i \in \Omega} b^i \). Consider an alternative sample space \( \bar{\Omega} \), in which each element of \( i \in \Omega \) is repeated \( \frac{a^i}{b} \) \( B \) times, noting that this quantity is a positive integer. This repetition defines a one-to-many correspondence from \( \Omega \) to \( \bar{\Omega} \), \( \rho \), which can be used to define asset and security values. Noting that \( \rho \) is invertible, we can define the asset values by \( \bar{v}_j = v_{\rho^{-1}(j)} \), and define \( \bar{s} \) and \( \bar{\eta} \) using the same transformation.

This repetition can also be used to defined a Markov congruent embedding between \( \mathcal{P}(\Omega) \) and \( \mathcal{P}(\bar{\Omega}) \). For each \( p \in \mathcal{P}(\Omega) \),

\[
\bar{p}^i = (\Pi p)^i = \frac{b^{\rho^{-1}(j)}}{\rho a^{\rho^{-1}(j)}} p^{\rho^{-1}(j)}.
\]

It follows that \( \bar{q} = \Pi q \) is the uniform distribution (\( \bar{q}^j = B^{-1} \) for each \( j \in \bar{\Omega} \)).

Now consider the utility in the modified security design problem (on sample space \( \bar{\Omega} \), with asset values \( \bar{v} \), security \( \bar{s} \), and probability distribution \( \bar{q} \)). By the invariance of the divergence, \( \partial_i D(p(\eta, q)) \) is invariant, and it immediately follows that

\[
\bar{p}^i(\bar{\eta}, \bar{q}) = (\Pi p(\eta, q))^i = \frac{b^{\rho^{-1}(j)}}{\rho a^{\rho^{-1}(j)}} p^i(\eta, q) = \frac{\bar{q}^j}{q^j} p^i(\eta, q).
\]
From the invariance of $D$, it follows that the utility in the modified security design problem is identical to the utility in the original security design problem.

Now suppose that there exists an optimal security design, some $\bar{v}_j \geq \bar{v}_i$ with $\bar{\eta}^*_i > \bar{\eta}^*_j$, which could only occur if there same condition held for some $i,j \in \Omega$ in the original problem. Consider an alternative security design which switches $\eta_i$ and $\eta_j$, leaving all other points unchanged. Note that this switch is feasible under the limited liability constraints. Because $\bar{q}_i = \bar{q}_j$, one can imagine that this perturbation also “switches” $\bar{\eta}^*$ and $\bar{\eta}^*$. It follows by the invariance of $\partial_i D$ that simply switching $\bar{p}_i(\bar{\eta}^*, \bar{q})$ and $\bar{p}_j(\bar{\eta}^*, \bar{q})$ will be the solution to the moral hazard FOC (equation (G.1)).

By the invariance of $D$, the seller’s indirect utility in the moral hazard problem will be unchanged by this switch. The price received from the buyer, on the other hand, will change by

$$\beta_b(s^*_j - s^*_i)(\bar{p}_i(\bar{\eta}^*, \bar{q}) - \bar{p}_j(\bar{\eta}^*, \bar{q})).$$

By the strict convexity of $D$, because $\bar{\eta}^*_i > \bar{\eta}^*_j$, then $\bar{p}_i(\bar{\eta}^*, \bar{q}) > \bar{p}_j(\bar{\eta}^*, \bar{q})$. By assumption,

$$s^*_j - s^*_i = \bar{v}_j - \bar{v}_i + \beta^{-1}_s(\bar{\eta}^*_i - \bar{\eta}^*_j) > 0,$$

and therefore this switch will strictly increase utility.

I have show that, in the modified security design problem, the optimal security design must feature a monotone retained tranche, $\bar{\eta}$. By the definition of monotonicity, if $v_i = v_j$, it must be the case that both $\eta_i \geq \eta_j$ and $\eta_j \leq \eta_i$, and therefore $\eta_j = \eta_i$ and $s_j = s_i$. It follows that, for all $i \in \Omega$, the optimal security in the modified problem assigns the same security value $s_j$ for all $j \in \rho(i)$. Therefore, there exists a security design in the original problem that would be mapped, using $\rho$, to the optimal security design in the modified problem. Every security design in the original problem can be mapped to a security design in the modified problem, and the utility that each security design achieves in the original problem is that same as the corresponding security in the modified problem. It follows that the security design which corresponds to the optimal security design in the modified problem must be the optimal security in the original problem. Because the optimal security design in the modified problem has a monotone retained tranche, its corresponding security design in the original problem also has a monotone retained tranche.

It follows that, if $q$ is rational, the optimal security design in the original problem must feature a monotone retained tranche. Now suppose $q$ is not rational. The rationals are dense in the reals, and therefore there exists a sequence of rational $\tilde{q}$ whose limit is $q$. If $v_j \geq v_i$ and $\tilde{\eta}^*_i > \tilde{\eta}^*_j$, then for each rational $\tilde{q}$ in this sequence, there is a strict welfare improvement from this switch. By the continuity of the utility in the security design problem, in the limit, utility is at least weakly higher from this switch, and therefore $\square$
Lemma 4. In the non-parametric security design problem with an \( \alpha \)-divergence, if \( \eta^*_i > 0 \) (\( s^*_i < v_i \)) for some \( i \in \Omega \) and \( \eta^*_j < \beta s_j \) (\( s^*_j > 0 \)) for some \( j \in \Omega \), it must be the case that

\[
\left( 1 + \kappa - \kappa \frac{1 + \alpha}{2} \right) (\eta^*_j - \eta^*_i) \geq \beta (v_j - v_i),
\]

or equivalently that

\[
-(1 + \kappa - \kappa \frac{1 + \alpha}{2}) (s^*_j - s^*_i) \geq \frac{1 + \alpha}{2} (v_j - v_i).
\]

Proof. This is essentially the argument in the text, but I will prove it without referencing the second derivatives of \( f \), because of the issue that, when \( \alpha \in (-3, -1) \), \( p^i(\eta^*) = 0 \) is possible and \( f''(0) \) is infinite.

The seller’s moral hazard problem, in Lagrangian form, is

\[
\phi(\eta) = \max_{p \in \mathbb{R}^{N+1}_+} \min_{\lambda, \nu \geq 0} \sum_i p^i \eta_i - \theta \sum_i q^i f\left( \frac{p^i}{q^i} \right) + \xi (1 - \sum_i p^i) + \sum_i \nu^i p^i,
\]

where \( \xi \) is the multiplier on the constraint that probabilities sum to one and \( \nu^i \) is the multiplier on the constraint that \( p^i \geq 0 \). The KKT conditions are necessary and sufficient due to the convexity of \( f \). The FOC with respect to \( p^i \) is

\[
\eta_i - \theta f'(\frac{p^i}{q^i}) - \xi + \nu_i = 0.
\]

For \( i = 0 \),

\[
\xi = \nu_0 - \theta f'(\frac{p^0}{q^0}).
\]

Recall the definition of the cost function and its derivative:

\[
f(u) = \begin{cases} 
\frac{4}{1-\alpha^2}(1 - u^{\frac{1}{2}(1-\alpha)} + \frac{1}{2}(1 - \alpha)(u - 1)), & \alpha \neq 1, -1 \\
u \ln u - u + 1 & \alpha = -1, \\
- \ln u + u - 1 & \alpha = 1,
\end{cases}
\]

\[
f'(u) = \begin{cases} 
\frac{2}{1+\alpha}(1 - u^{-\frac{1}{2}(1+\alpha)}) & \alpha \neq -1, \\
\ln u & \alpha = 1.
\end{cases}
\]

If \( \alpha < -1 \),

\[
\lim_{u \to 0^+} f(u) = \frac{2}{1-\alpha} = f(0)
\]

\[
\lim_{u \to 0^+} f'(u) = \frac{2}{1+\alpha} = f'(0),
\]
implying that it is feasible for the agent to set \( p^i = 0 \) for some \( i \in \Omega \). For \( \alpha > -1 \),

\[
\lim_{u \to 0^+} f'(u) = -\infty,
\]

implying that it will never be optimal for an agent to set \( p^i = 0 \).

I will begin by showing that, under the optimal security design, the multipliers \( \nu_i^* = 0 \) for all \( i \in \Omega \). There exists a \( j \in \Omega \) such that \( p^j(\eta^*) > 0 \). Using this \( j \), the FOC of the moral hazard problem can be written as

\[
\eta_i^* - \eta_j^* + \theta f'(\frac{p^j(\eta^*)}{q^j}) - \theta f'(\frac{p^j(\eta^*)}{q^i}) + \nu_i^* = 0.
\]

By the complementary slackness condition and the convexity of \( f \), if \( \eta_i^* \geq \eta_j^* \), then \( p^j(\eta^*) > 0 \). Note that \( \Omega \) is finite, and therefore \( \eta_i^* \) takes on a finite set of values. It follows that there exists an \( \bar{\eta}(\eta^*) \) such that, for all \( \eta_i^* \geq \bar{\eta}(\eta^*) \), \( p^j(\eta^*) > 0 \), and for all \( \eta_i^* < \bar{\eta}(\eta^*) \), \( p^j(\eta^*) = 0 \). If \( \alpha > -1 \), and \( p^j(\eta^*) > 0 \) always, then \( \bar{\eta}(\eta^*) = 0 \).

Suppose that there exists an \( \eta_i^* > 0 \) such that \( p^j(\eta^*) = 0 \), implying that \( \bar{\eta}(\eta^*) > 0 \) and \( i > 0 \). Consider an alternative security design in which \( \eta \) equals \( \eta^* \), except that \( \eta_i = 0 \). Recall the first-order condition of the seller’s moral hazard problem:

\[
\eta_j - \theta f'(\frac{p^j}{q^j}) + \theta f'(\frac{p^0}{q^0}) - \nu_0 + \nu_j = 0.
\]

Conjecture that, under the alternative security design, for all \( j \neq i \), \( \nu_j = \nu_j^* \) and \( p^j = p^j(\eta^*) \), and that \( \nu_i = \eta_i^* \) and \( p^i = p^i(\eta^*) = 0 \). The KKT conditions are satisfied, and by their sufficiency the conjecture is verified. It follows that this alternative security design would result in the same probability distribution, \( p(\eta^*) \). Because \( p^i(\eta^*) = 0 \), the new security and retained tranche designs have the same expected value as under the optimal designs. It follows that the alternative design delivers the same utility, and is also optimal. It is therefore without loss of generality to consider optimal security designs such that, if \( p^i(\eta^*) = 0 \), then \( \eta_i^* = 0 \). For such designs, by the first-order conditions in the seller’s moral hazard problem, it must be the case that if \( \nu_i^* > 0 \), then \( \nu_0^* = \nu_i^* > 0 \), by the fact that \( \eta_0^* = 0 \).

Consider such a security design, and suppose that \( \nu_0^* > 0 \), implying that \( p^i(\eta^*) = 0 \) if \( \eta_i^* = 0 \). Now consider a perturbation to the optimal security design, which lowers all \( \eta_i \) with \( \eta_i^* > 0 \) by an amount \( \epsilon \). By assumption, if \( \eta_i^* > 0 \), then \( p^i(\eta^*) > 0 \). Conjecture that this perturbation lowers \( \nu_0 \), and all other \( \nu_i = \nu_0 \) associated with other \( \eta_i^* = 0 \), by an amount \( \epsilon \), leaving all probabilities unchanged. Inspecting the first-order conditions of the seller’s moral hazard problem verifies the conjecture. It follows that this perturbation reduces the expected value of the retained tranche, increasing the expected value of the security, while holding the probability distribution unchanged. It therefore raises welfare in the security design problem.
It immediately follows that \( \nu_0^* = 0 \), and therefore \( \nu_i^* = 0 \) for all \( i \in \Omega \), regardless of whether \( p^i(\eta^*) = 0 \) or not. The solution to the moral hazard problem is “just interior,” in the sense that, for all \( i, j \in \Omega \),

\[
\eta_i^* - \eta_j^* + \theta f'(\frac{p^i(\eta^*)}{q^j}) - \theta f'(\frac{p^j(\eta^*)}{q^i}) = 0.
\] (G.2)

It immediately follows that \( p^i(\eta^*) = 0 \) can hold only if \( \eta_i^* = 0 \).

Consider the \( i, j \in \Omega \) given in the proposition. By the fact that \( \eta_i^* > 0 \), \( p^i(\eta^*) > 0 \), but it is possible (if \( \eta_j^* = 0 \)) that \( p^j(\eta^*) = 0 \). Now consider a perturbation that decreases \( \eta_i \) by \( \theta f'(\frac{p^i(\eta^*)}{q^j}) - \theta f'(\frac{\theta^{-1} \epsilon + p^i(\eta^*)}{q^j}) \), and increases \( \eta_j \) by \( \theta f'(\frac{\theta^{-1} \epsilon + p^j(\eta^*)}{q^i}) - \theta f'(\frac{p^j(\eta^*)}{q^j}) \). Note that, by the convexity of \( f \), both of these expressions are positive. By the second differentiability of \( f \), the decrease in \( \eta_i \) is on the order of \( \epsilon \), and therefore feasible for small \( \epsilon \). This also applies to the increase in \( \eta_j \), if \( p^j(\eta^*) > 0 \). If \( p^j(\eta^*) = 0 \), the increase in \( \eta_j \) would be

\[
\theta f'(\frac{\theta^{-1} \epsilon}{q^j}) - \theta f'(0) = -\frac{2\theta}{1 + \alpha} \left( \frac{\theta^{-1} \epsilon}{q^j} \right)^{-\frac{1}{2}(1+\alpha)}.
\]

If \( p^j(\eta^*) = 0 \), it must be the case that \( \alpha < -1 \), and therefore this change is also small (and therefore feasible) for small \( \epsilon \).

Consider the first-order conditions of the seller’s moral hazard problem, and conjecture that the perturbation increases \( p^j \) by \( \theta^{-1} \epsilon \), decreases \( p^i \) by the same amount, and leaves all other probabilities, and all multipliers (aside from possibly \( \nu_j \)) unchanged. The first order condition, for any \( k \notin \{i, j\} \), is still

\[
\eta_k - \theta f'(\frac{p^k(\eta^*)}{q^k}) - \xi^* = 0,
\]

and therefore unchanged. The first order conditions for \( p^j \) and \( p^i \) are satisfied by construction. Because \( p^j \) is increasing and \( p^j(\eta^*) > 0 \), the positivity constraints are satisfied (for small enough \( \epsilon \)), and because they increase/decrease by the same amounts, the add-to-one constraint is still satisfied. By the sufficiency of the KKT conditions for this problem, it follows that the conjecture is verified.

The utility before the perturbation is

\[
U_0 = \beta_0 \sum_k p^k(\eta^*)v_k - \kappa \sum_k p^k(\eta^*)\eta_k^* - \theta \sum_k q^k f(\frac{p^k(\eta^*)}{q^k}),
\]
and afterwards is

\[ U_0 + \beta_0 \theta^{-1} \epsilon (v_j - v_i) + \kappa p^i(\eta^*) \eta_i^* + \kappa p^j(\eta^*) \eta_j^* - \]

\[ \kappa(p^i(\eta^*) - \theta^{-1} \epsilon)(\eta_i^*) + \theta f'(\frac{-\theta^{-1} \epsilon + p^i(\eta^*)}{q^i}) - \theta f'(\frac{p^i(\eta^*)}{q^i}) - \]

\[ \kappa(p^j(\eta^*) + \theta^{-1} \epsilon)(\eta_j^*) + \theta f'(\frac{\theta^{-1} \epsilon + p^j(\eta^*)}{q^j}) - \theta f'(\frac{p^j(\eta^*)}{q^j}) + \]

\[ \theta q^i \frac{1}{\epsilon} \left( f \left( \frac{p^i(\eta^*)}{q^i} \right) - f \left( \frac{-\theta^{-1} \epsilon + p^i(\eta^*)}{q^i} \right) \right) + \]

\[ \theta q^i \frac{1}{\epsilon} \left( f \left( \frac{p^j(\eta^*)}{q^j} \right) - f \left( \frac{-\theta^{-1} \epsilon + p^j(\eta^*)}{q^j} \right) \right) = U_1. \]

We can rewrite this as

\[ \frac{U_1 - U_0}{\epsilon} = \beta_0 \theta^{-1} (v_j - v_i) - \kappa p^i(\eta^*) \frac{\theta}{\epsilon} \left( f' \left( \frac{-\theta^{-1} \epsilon + p^i(\eta^*)}{q^i} \right) - f' \left( \frac{p^i(\eta^*)}{q^i} \right) \right) - \]

\[ \kappa(\theta^{-1} \eta_j^* + f' \left( \frac{-\theta^{-1} \epsilon + p^j(\eta^*)}{q^j} \right) - f' \left( \frac{p^j(\eta^*)}{q^j} \right) \right) - \]

\[ \kappa p^i(\eta^*) \frac{\theta}{\epsilon} \left( f' \left( \frac{-\theta^{-1} \epsilon + p^i(\eta^*)}{q^i} \right) - f' \left( \frac{p^i(\eta^*)}{q^i} \right) \right) + \]

\[ \kappa(\theta^{-1} \eta_i^* + f' \left( \frac{-\theta^{-1} \epsilon + p^i(\eta^*)}{q^i} \right) - f' \left( \frac{p^i(\eta^*)}{q^i} \right) \right) + \]

\[ \theta q^j \frac{1}{\epsilon} \left( f \left( \frac{p^j(\eta^*)}{q^j} \right) - f \left( \frac{-\theta^{-1} \epsilon + p^j(\eta^*)}{q^j} \right) \right) + \]

\[ \theta q^j \frac{1}{\epsilon} \left( f \left( \frac{p^i(\eta^*)}{q^i} \right) - f \left( \frac{-\theta^{-1} \epsilon + p^i(\eta^*)}{q^i} \right) \right) \].

Suppose that \( p^j(\eta^*) = 0 \). Taking the limit as \( \epsilon \to 0^+ \),

\[ \lim_{\epsilon \to 0^+} \frac{U_1 - U_0}{\epsilon} = \beta_0 \theta^{-1} (v_j - v_i) + \kappa \frac{p^i(\eta^*)}{q^i} f'' \left( \frac{p^i(\eta^*)}{q^i} \right) \]

\[ + \kappa \theta^{-1} (\eta_i^* - \eta_j^*) + \frac{f'(\frac{p^i(\eta^*)}{q^i}) - f'(\frac{p^j(\eta^*)}{q^j})}{\epsilon} \].
If \( p^j(\eta^*) > 0 \), a similar limit holds:

\[
\lim_{\epsilon \to 0^+} \frac{U_1 - U_0}{\epsilon} = \beta_b \theta^{-1}(v_j - v_i) - \frac{1 + \alpha}{2} \kappa \theta^{-1} + \frac{\kappa \theta^{-1}(\eta^*_i - \eta^*_j) + f'(\frac{p^j(\eta^*)}{q^i}) - f'(\frac{p^i(\eta^*)}{q^j})}{\epsilon}.
\]

Using the fact that, if \( u > 0 \),

\[
u f''(u) = 1 + \frac{\alpha}{2} f'(u),
\]

we can rewrite this as

\[
\lim_{\epsilon \to 0^+} \frac{U_1 - U_0}{\epsilon} = \beta_b \theta^{-1}(v_j - v_i) + \frac{1 + \alpha}{2} \kappa \theta^{-1} + \frac{\kappa \theta^{-1}(\eta^*_i - \eta^*_j) + f'(\frac{p^j(\eta^*)}{q^i}) - f'(\frac{p^i(\eta^*)}{q^j})}{\epsilon}.
\]

Note that, because

\[
f'(0) = \frac{2}{1 + \alpha},
\]

this holds regardless of whether \( p^i(\eta^*) = 0 \) or not. By the “just-interior” assumption,

\[
f'(\frac{p^j(\eta^*)}{q^i}) - f'(\frac{p^j(\eta^*)}{q^i}) = \theta^{-1}(\eta^*_i - \eta^*_j),
\]

and therefore,

\[
\lim_{\epsilon \to 0^+} \frac{U_1 - U_0}{\epsilon} = \beta_b \theta^{-1}(v_j - v_i) - \frac{1 + \alpha}{2} \kappa \theta^{-1} + \frac{\kappa \theta^{-1}(\eta^*_i - \eta^*_j) + \theta^{-1}(\eta^*_i - \eta^*_j)}{\epsilon}.
\]

This simplifies to

\[
\lim_{\epsilon \to 0^+} \frac{U_1 - U_0}{\epsilon} = \theta^{-1} \beta_b (v_j - v_i) + \theta^{-1}(\eta^*_i - \eta^*_j)(1 + \kappa \frac{1 - \alpha}{2}).
\]

The perturbation will deliver a welfare improvement unless

\[
\theta^{-1} \beta_b (v_j - v_i) + \theta^{-1}(\eta^*_i - \eta^*_j)(1 + \kappa \frac{1 - \alpha}{2}) \leq 0,
\]
which is equivalent to
\[
\left(\frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa}\right) (\eta_j^* - \eta_i^*) \geq \beta_s (v_j - v_i).
\]

To finish the proof, substitute in the definition of the retained tranche:
\[
\left(\frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa}\right) (v_j - s_j^* - v_i + s_i^*) \geq (v_j - v_i),
\]
and therefore
\[
-(1 + \kappa - \kappa^{1+\alpha}) \frac{1}{2} (s_j^* - s_i^*) \geq \frac{1}{2} (v_j - v_i).
\]

\[\square\]

G.5. **Proof of proposition 3.** The monotonicity of the retained tranche, for these cost functions, is proven in lemma 3. The proof divides into three cases: \(\alpha < -1\), \(\alpha \in [-1, 1 + \frac{2}{\kappa}]\), and \(\alpha > 1 + \frac{2}{\kappa}\). I also separately prove that in all of these cases, when \(v_N\) is sufficiently large, \(\bar{v} < v_N\), and that when \(\alpha \geq -3\), \(\bar{v} > 0\).

G.5.1. \(\alpha < -1\). If the optimal security is selling everything (\(\eta_i^* = 0\)), the proposition is satisfied by setting \(\bar{v} = v_N\). If the optimal security were selling nothing (which I will prove it is not, below), the proposition would be satisfied (aside from having negative \(\bar{v}\)) if, for all \(i \in \Omega\),
\[
\frac{-\kappa (1 + \alpha)}{2 + \kappa (1 - \alpha)} (v_j - \bar{v}) + \bar{v} \leq 0.
\]
Because \(\alpha < -1\),
\[
0 < \frac{-\kappa (1 + \alpha)}{2 + \kappa (1 - \alpha)} < 1,
\]
and therefore
\[
\bar{v} = \frac{\kappa (1 + \alpha)}{2 + 2\kappa} v_N < 0
\]
is sufficient.

Now assume that the optimal security design is not selling nothing or selling everything, but it is still the case that, for all \(i \in \Omega\), \(\eta_i^* \in \{0, \beta_s v_i\}\). By monotonicity, there must be some \(\eta_j^* = 0\) and \(\eta_i^* = \beta_s v_i\) such that \(i > j\), and therefore \(v_i \geq v_j\). Lemma 4 applies:
\[
\left(\frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa}\right) \beta_s v_i \leq \beta_s (v_i - v_j).
\]
Because \(\alpha < -1\),
\[
\left(\frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa}\right) > 1,
\]
and therefore this is a contradiction. Therefore, if the security design is not selling everything or nothing, there must be some \(j \in \Omega\) with \(\eta_j^* \in (0, \beta_s v_j)\).
In this case, for all \( i > j, v_i \geq v_j \) and (by monotonicity) \( \eta^*_i \geq \eta^*_j > 0 \), and lemma 4 applies:

\[
\left( \frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa^{1+\alpha}} \right)(\eta^*_i - \eta^*_j) \leq \beta_s(v_i - v_j).
\]

Because \( \alpha < -1 \),

\[
\left( \frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa^{1+\alpha}} \right)(\eta^*_i - \eta^*_j) \geq (\eta^*_i - \eta^*_j),
\]

and therefore

\[
\eta^*_i \leq \beta_s v_i + (\eta^*_j - \beta_s v_j) < \beta_s v_i.
\]

That is, if \( \eta^*_j \) is interior, then \( \eta^*_i \) is also interior for all \( i > j \). For any \( i, j \) such that \( \eta^*_i \) and \( \eta^*_j \) are both interior, lemma 4 and its “reverse” both apply, and we must have

\[
\left( \frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa^{1+\alpha}} \right)(\eta^*_i - \eta^*_j) = \beta_s(v_i - v_j).
\]

It follows that there is some index \( \bar{i} \) (which could be zero) such that, for all \( i > \bar{i}, \eta^*_i \in (0, \beta_s v_i) \),

and for all \( i \leq \bar{i}, \eta^*_i \in \{0, \beta_s v_i\} \). Define

\[
\bar{v} = v_i - \frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa^{1+\alpha}} \beta^{-1} \eta^*_i.
\]

It follows that, for all \( j \geq \bar{i} \),

\[
\left( \frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa^{1+\alpha}} \right)\eta^*_j = \beta_s(v_j - \bar{v}),
\]

or equivalently

\[
\beta_s^*(\frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa^{1+\alpha}}) = \left( -\frac{\kappa^{1+\alpha}}{1 + \kappa} \right)v_j + \bar{v}.
\]

Solving,

\[
\beta_s^* = \frac{-\kappa(1 + \alpha)}{2 + \kappa(1 - \alpha)} v_j + \frac{1 + \kappa}{1 + \kappa - \kappa^{1+\alpha}} \bar{v} = \frac{-\kappa(1 + \alpha)}{2 + \kappa(1 - \alpha)} (v_j - \bar{v}) + \bar{v}.
\]

Note that, because \( \alpha \leq 1 + \frac{2}{\kappa}, \bar{v} < v_i \).

Suppose that \( \eta^*_j = 0 \) for some \( j \in \Omega \) with \( v_j > 0 \). In this case, applying lemma 4 with \( \bar{i} \) in the place of \( i \),

\[
\left( \frac{1 + \kappa - \kappa^{1+\alpha}}{1 + \kappa^{1+\alpha}} \right)\eta^*_i \leq \beta_s(v_i - v_j),
\]

or

\[
v_j \leq \bar{v},
\]
which can only occur if \( \bar{v} \geq 0 \). Along the same lines, suppose \( \eta_i^* = \beta_s v_i > 0 \) for some \( i \in (0, \bar{i}) \). Applying lemma 4 with \( \bar{i} \) in the place of \( j \),

\[
\left( 1 + \kappa - \frac{\kappa^{1+\alpha}}{2} \right) (\eta_i^* - \beta_s v_i) \geq \beta_s (v_i - v_i),
\]

or

\[
v_i \frac{\kappa^{1+\alpha}}{1 + \kappa} \geq \bar{v}.
\]

Because \( \alpha < -1 \), this can only occur if \( \bar{v} < 0 \).

It follows that these two cases are mutually exclusive: either \( \bar{v} < 0 \) and there can exist \( \eta_i^* = \beta_s v_i > 0 \), or \( \bar{v} \geq 0 \) and there can exist \( \eta_i^* = 0 \) with \( v_i > 0 \). In the first case, if \( s_i^* = 0 \), then

\[
v_i \frac{\kappa^{1+\alpha}}{1 + \kappa} \geq \bar{v},
\]

implying that

\[
\left( -\frac{\kappa^{1+\alpha}}{1 + \kappa} \right) v_j + \bar{v} \leq 0,
\]

and therefore

\[
s_i^* = \max \left( -\frac{\kappa(1 + \alpha)}{2 + \kappa(1 - \alpha)} (v_i - \bar{v}) + \bar{v}, 0 \right)
\]

for all \( i \), proving the proposition. In the second case, \( \bar{v} \geq 0 \), and if \( v_i < \bar{v} \), then

\[
-\frac{\kappa(1 + \alpha)}{2 + \kappa(1 - \alpha)} (v_i - \bar{v}) + \bar{v} > v_i,
\]

because \( \alpha < -1 \) implies that

\[
0 < -\frac{\kappa(1 + \alpha)}{2 + \kappa(1 - \alpha)} < 1.
\]

It follows that the security cannot be interior, and therefore it must be the case that \( s_i^* = v_i \), consistent with the proposition.

Lastly, I will show that \( \bar{v} \geq 0 \), strictly if \( \alpha > -3 \). As shown above, if \( \bar{v} \leq 0 \), then \( \eta_i^* > 0 \) for all \( i \) such that \( v_i > 0 \). By the convexity of \( f \)-divergences, for all \( i \) such that \( \eta_i = v_i = 0 \), under an optimal or sub-optimal security design, the seller will set \( \frac{p_i}{q_i} = \frac{0}{\theta} \). It follows by the invariance property of \( f \)-divergences that it is without loss of generality to assume there is only one state with \( v_i = 0 \), which is \( i = 0 \).

Consider a perturbation which decreases all \( \eta_i > 0 \) (all \( i > 0 \)) by an amount

\[
\Delta \eta_i = \theta f' \left( \frac{p^i(\eta^*)}{q^i} \right) - \theta f' \left( \frac{-\theta^{-1} \epsilon + p^i(\eta^*)}{q^i} \right) - \theta f' \left( \frac{p^0(\eta^*)}{q^0} \right) + \theta f' \left( \frac{-\theta^{-1} N \epsilon + p^0(\eta^*)}{q^0} \right).
\]
By the strict convexity of $f$, $\Delta \eta_i > 0$. If $p^0(\eta^*) > 0$, then

$$\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta \eta_i = \frac{1}{q^i} f''(p^i(\eta^*)) + \frac{N}{q^0} f''(p^0(\eta^*)).$$

If $p^0(\eta^*) = 0$ and $\alpha < -3$,

$$\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta \eta_i = \frac{1}{q^i} f''(p^i(\eta^*)) - \lim_{\epsilon \to 0^+} \frac{2}{1 + \alpha} \epsilon^{-1} \left(\frac{\theta^{-1} N}{q^0}\right)^{-\frac{1}{2}(1+\alpha)}$$

$$= \frac{1}{q^i} f''(p^i(\eta^*)).$$

If $p^0(\eta^*) = 0$ and $\alpha = -3$,

$$\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta \eta_i = \frac{1}{q^i} f''(p^i(\eta^*)) - \frac{2}{1 + \alpha} \left(\frac{\theta^{-1} N}{q^0}\right)^{-\frac{1}{2}(1+\alpha)} > 0.$$

If $p^0(\eta^*) = 0$ and $\alpha \in (-3, -1)$, then $\frac{1}{2}(1 + \alpha) \in (-1, 0)$, and

$$\lim_{\epsilon \to 0^+} \epsilon^{\frac{1}{2}(1+\alpha)} \Delta \eta_i = -\frac{2}{1 + \alpha} \left(\frac{\theta^{-1} N}{q^0}\right)^{-\frac{1}{2}(1+\alpha)} > 0.$$

It follows in all cases that $\lim_{\epsilon \to 0^+} \Delta \eta_i = 0$, and therefore the perturbation is feasible for sufficiently small $\epsilon$.

By equation (G.2) in the proof of lemma 4, one can immediately conjecture and verify that this perturbation reduces all $p^i$, for $i > 0$, by $\theta^{-1} \epsilon$, and increases $p^0$ by $N \theta^{-1} \epsilon$. The indirect effect is therefore

$$-\beta_h \theta^{-1} \epsilon \sum_{i \in \Omega} s_i^*,$$

and the direct effect is

$$\kappa \sum_{i \in \Omega \setminus \{0\}} p^i(\eta^*) \Delta \eta_i.$$

If $\alpha \in (-3, -1)$ and $p^0(\eta^*) = 0$, we can scale both effects by $\epsilon^{\frac{1}{2}(1+\alpha)}$, noting that $\frac{1}{2}(1 + \alpha) \in (-1, 0)$, and take the limit as $\epsilon \to 0^+$. The indirect effect converges to zero, the direct effect is strictly positive, and therefore this is welfare improving, and we must have $\tilde{v} > 0$. If $\alpha \leq -3$ and $p^0(\eta^*) = 0$, or if $p^0(\eta^*) > 0$, we can scale both effects by $\epsilon^{-1}$ and take the limit as $\epsilon \to 0^+$.

When $p^0(\eta^*) > 0$, we find that

$$\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta U = -\beta_h \theta^{-1} \sum_{i \in \Omega} s_i^* + \frac{N(1 - p^0(\eta^*))}{q^0} f''(p^0(\eta^*))$$

$$+ \kappa \sum_{i \in \Omega} \frac{p^i(\eta^*)}{q^i} f''(p^i(\eta^*)).$$
By the fact that (for $u > 0$),
\[ u f''(u) = 1 - \frac{1 + \alpha}{2} f'(u) \]
and
\[ f'(\frac{p^i(\eta^*)}{q^i}) - f'(\frac{p^0(\eta^*)}{q^0}) = \theta^{-1} \eta^*_i, \]
we can rewrite this as
\[
\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta U = -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* + \kappa \frac{N}{q^0} f''(\frac{p^0(\eta^*)}{q^0})
\]
\[
- \frac{1 + \alpha}{2} \kappa \theta^{-1} \sum_{i \in \Omega} \eta^*_i
\]
\[
> -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* + \frac{1 + \alpha}{2} \kappa \theta^{-1} \sum_{i \in \Omega} \eta^*_i.
\]
If $\bar{v} \leq 0$, we must have
\[ s_i^* \leq -\frac{\kappa(1 + \alpha)}{2 + \kappa(1 - \alpha)} v_i, \]
and therefore
\[ \beta_b s_i^* \leq -\frac{\kappa(1 + \alpha)}{2} \eta_i^*. \]
It follows that the perturbation would improve welfare, and therefore $\bar{v} > 0$ if $p^0(\eta^*) > 0$.

Applying similar logic, if $\alpha = -3$ and $p^0(\eta^*) = 0$,
\[
\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta U = -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* - \frac{2\kappa}{1 + \alpha} \left( \frac{\theta^{-1} N}{q^0} \right)^{-\frac{1}{2}(1 + \alpha)}
\]
\[
+ \kappa \sum_{i \in \Omega} \frac{p^i(\eta^*)}{q^i} f''(\frac{p^i(\eta^*)}{q^i}).
\]
\[
> -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* + \kappa \sum_{i \in \Omega} \frac{p^i(\eta^*)}{q^i} f''(\frac{p^i(\eta^*)}{q^i}),
\]
where the inequality holds by $\alpha < -1$. Using
\[ u f''(u) = 1 - \frac{1 + \alpha}{2} f'(u), \]
\[
\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta U > -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* - \kappa \frac{1 + \alpha}{2} \sum_{i \in \Omega} \left( \frac{2}{1 + \alpha} - f'(p^i(\eta^*)) \right)
\]
\[
= -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* - \kappa \frac{1 + \alpha}{2} \sum_{i \in \Omega} \left( f'(0) - f'(p^i(\eta^*)) \right)
\]
\[
= -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* + \kappa \theta^{-1} \frac{1 + \alpha}{2} \sum_{i \in \Omega} \eta_i^*.
\]

It again follows that \( \bar{v} > 0 \).

Finally, consider the case when \( \alpha < -3 \) and \( p^0(\eta^*) = 0 \):
\[
\lim_{\epsilon \to 0^+} \epsilon^{-1} \Delta U = -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* + \kappa \sum_{i \in \Omega} \frac{p^i(\eta^*)}{q^i} f''(p^i(\eta^*))
\]
\[
= -\beta_b \theta^{-1} \sum_{i \in \Omega} s_i^* + \kappa \theta^{-1} \frac{1 + \alpha}{2} \sum_{i \in \Omega} \eta_i^*.
\]

This is identical to the previous case, with an equality instead of an inequality. It follows that \( \bar{v} \geq 0 \) always, strictly if \( \alpha \geq -3 \).

G.5.2. \( \alpha \in [-1, 1 + \frac{2}{\kappa}] \). In these cases, the solution to the seller’s moral hazard problem is guaranteed to be interior. The Lagrangian version of the security design problem is
\[
\mathcal{L}(\eta, \lambda, \omega) = \max_{\eta} \min_{\lambda \geq 0, \omega \geq 0} \beta_b \sum_{i > 0} p^i(\eta) s_i(\eta) + \phi(\eta) + \kappa \sum_{i > 0} \lambda_i \eta_i + \kappa \sum_{i > 0} \omega_i (\beta_i v_i - \eta_i),
\]
where \( \lambda \) and \( \omega \) are the multipliers on the constraints that \( \eta_i \geq 0 \) and \( \eta_i \leq \beta_i v_i \), respectively. The constraints are affine, and therefore the KKT conditions are necessary. The FOC with respect to \( \eta_i, i > 0 \), is
\[
(\text{G.3}) \quad \beta_b s_j(\eta^*) = \kappa \sum_{i > 0} [p^j(\eta^*) - \lambda_i + \omega_i] \frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} |_{p=p(\eta^*)},
\]
using the envelope theorem (due to the interior solution),
\[
\partial_i \phi(\eta)|_{\eta=\eta^*} = p^i(\eta^*),
\]
and the result that \( \frac{dp^i}{d\eta} \) is the inverse of the Hessian (which is derived by differentiating this expression again and using the properties of convex conjugates, see Amari and Nagaoka [2007]).

For the \( \alpha \)-divergences, in the coordinate system parametrized by \( p^i, i \in \{1, \ldots, N\} \),
\[
\frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} |_{p=p(\eta^*)} = \delta_{ij} \frac{\theta f''(p^i(\eta^*))}{q^i} + \eta_i \eta_j \frac{\theta f''(p^0(\eta^*))}{q^0},
\]
where \( \iota \) is a vector of ones and \( \delta_{ij} \) is the Kronecker delta function. As noted in the text,

\[ u f''(u) = 1 - \frac{1 + \alpha}{2} f'(u), \]

and therefore

\[
\frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} = \delta_{ij} \frac{\theta}{p^i} + \iota_i \iota_j \frac{\theta}{p^0} - \frac{1 + \alpha}{2} \left[ \delta_{ij} \frac{\theta f'(p^i)}{p^i} + \frac{\theta f'(p^0)}{p^0} \right].
\]

By equation (G.2) in the proof of lemma 4, for all security designs, the seller’s first-order condition is

\[ \eta_i - \theta f'(\frac{p^i(\eta)}{q^i}) + \theta f'(\frac{p^0(\eta)}{q^0}) = 0. \]

Therefore,

\[
\frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} \bigg|_{p=p(\eta^*)} = \theta(1 - \frac{1 + \alpha}{2} f'(\frac{p^0(\eta^*)}{q^0})) g_{ij}(p(\eta^*))
\]

\[ - \delta_{ij} \frac{1 + \alpha}{2} \frac{\eta_i^*}{p^i(\eta^*)}. \]

By the definition of the f-divergence,

\[
1 - \frac{1 + \alpha}{2} f'(\frac{p^0(\eta^*)}{q^0}) = (\frac{p^0(\eta^*)}{q^0})^{-\frac{1}{2}(1+\alpha)}. \]

The security design FOC is

\[ \beta_s \eta_j(\eta^*) = \kappa \theta \left( \frac{p^0(\eta^*)}{q^0} \right)^{-\frac{1}{2}(1+\alpha)} \sum_{i>0} [p^i(\eta^*) - \lambda^i + \omega^j] g_{ij}(p(\eta^*))
\]

\[ - \kappa \frac{1 + \alpha}{2} \sum_{i>0} [p^i(\eta^*) - \lambda^i + \omega^i] \delta_{ij} \frac{\eta_i^*}{p^i(\eta^*)}. \]

This can be rewritten, using the complementary slackness conditions, as

\[ \beta_s(1 + \kappa) \eta_j(\eta^*) + (\kappa \frac{1 + \alpha}{2}) \eta_j^* = \kappa \theta \left( \frac{p^0(\eta^*)}{q^0} \right)^{-\frac{1}{2}(1+\alpha)} \sum_{i>0} [p^i(\eta^*) - \lambda^i + \omega^i] g_{ij}(p(\eta^*)) + \]

\[ - \kappa \frac{1 + \alpha}{2} \sum_{i>0} \omega^i \delta_{ij} \frac{\beta_s v_i}{p^i(\eta^*)}. \]
We can rewrite and therefore using the adding-up constraints on security designs.

Applying the first-order condition of the seller’s problem and the properties of \( f \)-divergences, 

\[
\alpha 
\]

Define

\[(G.4) \quad \bar{v} = E^{\mu^{(\eta^*)}}[s_j(\eta^*) + \beta^{-1}_b(1 + \frac{\alpha}{2})(\eta^*_j + \frac{\omega^j v_i}{p^j(\eta^*)})] + \beta^{-1}_b \kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)}.
\]

Note that, whenever \( \alpha \geq -1 \), 

\[
\bar{v} > 0,
\]

by the limited liability constraints. We can write our equation as 

\[
\beta_s(1 + \kappa)s_j(\eta^*) + (1 + \frac{\alpha}{2})\eta^*_j = \beta_b \bar{v} - \kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\lambda^j}{p^j(\eta^*)} + \frac{\omega^j}{p^j(\eta^*)} (\kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)} - \frac{1 + \alpha}{2} \beta_s v_j).
\]

Applying the first-order condition of the seller’s problem and the properties of \( f \)-divergences,

\[
\kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)} - \frac{1 + \alpha}{2} \beta_s v_j = \kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)},
\]

and therefore

\[
\beta_s(1 + \kappa)s_j(\eta^*) + (1 + \frac{\alpha}{2})\eta^*_j = \beta_b \bar{v} - \kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\lambda^j}{p^j(\eta^*)} + \kappa \theta \left( \frac{p^j(\eta^*)}{q^j} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\omega^j}{p^j(\eta^*)}.
\]

Using the adding-up constraints on security designs,

\[
\beta_b v_j + (1 + \frac{\alpha}{2} - 1 - \kappa)\eta^*_j = \beta_b \bar{v} - \kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\lambda^j}{p^j(\eta^*)} + \kappa \theta \left( \frac{p^j(\eta^*)}{q^j} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\omega^j}{p^j(\eta^*)}.
\]

We can rewrite \( \bar{v} \) as 

\[
\bar{v} = E^{\mu^{(\eta^*)}}[v_j + \beta^{-1}_b(1 + \frac{\alpha}{2} - 1 - \kappa)\eta^*_j + \beta^{-1}_b(1 + \frac{\alpha}{2}) \frac{\omega^j v_i}{p^j(\eta^*)}] + \beta^{-1}_b \kappa \theta \left( \frac{p^0(\eta^*)}{q_0^0} \right)^{-\frac{1}{2}(1+\alpha)}.
\]

When \( \alpha \leq 1 + \frac{2}{\kappa} \),

\[
\frac{1 + \alpha}{2} - 1 - \kappa \leq 0.
\]
It follows that
\[
\beta_s \kappa \left( \frac{1 + \alpha}{2} \right) v_j \leq \beta_b \tilde{v} - \kappa \theta \left( \frac{p^0(\eta^*)}{q^0} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\lambda^j}{p^j(\eta^*)} + \kappa \theta \left( \frac{p^j(\eta^*)}{q^j} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\omega^j}{p^j(\eta^*)} \leq \beta_b v_j.
\]
Recall, by the complementary slackness conditions, that \(\lambda^j \geq 0\) and \(\omega^j \geq 0\). If \(v_j < \tilde{v}\), we must have \(\lambda_j > 0\), and therefore \(s_j = v_j\) and \(\eta_j = 0\). If \(\kappa \left( \frac{1 + \alpha}{2} \right) \beta_s v_j > \beta_b \tilde{v}\), it must be the case that \(\omega_j > 0\) and therefore \(s_j = 0\) and \(\eta_j = \beta_s v_j\). Note that \(\beta_s \kappa \left( \frac{1 + \alpha}{2} \right) v_j > \beta_b \tilde{v}\) is equivalent to
\[
v_j > \frac{1 + \kappa}{\kappa} \frac{2}{1 + \alpha} \tilde{v},
\]
and when \(\alpha \in [-1, 1 + \frac{2}{\kappa}]\),
\[
\frac{1 + \kappa}{\kappa} \frac{2}{1 + \alpha} \geq 1.
\]
Suppose that \(v_j \in [\tilde{v}, \left( \frac{1 + \kappa}{\kappa} \frac{2}{1 + \alpha} \right) \tilde{v}]\), which is non-empty. If \(\eta_j^* = \beta_s v_j\), we would have \(\lambda_j = 0\), and
\[
\kappa \left( \frac{1 + \alpha}{2} \right) \beta_s v_j = \beta_b \tilde{v} + \kappa \theta \left( \frac{p^j(\eta^*)}{q^j} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\omega^j}{p^j(\eta^*)},
\]
which is possible only if \(v_j = \left( \frac{1 + \kappa}{\kappa} \frac{2}{1 + \alpha} \right) \tilde{v}\). Similarly, if \(\eta_j^* = 0\), we would have \(\omega_j = 0\) and
\[
\beta_b v_j = \beta_b \tilde{v} - \kappa \theta \left( \frac{p^0(\eta^*)}{q^0} \right)^{-\frac{1}{2}(1+\alpha)} \frac{\lambda^j}{p^j(\eta^*)},
\]
which is possible only if \(v_j = \tilde{v}\). It follows that if \( (\tilde{v}, \left( \frac{1 + \kappa}{\kappa} \frac{2}{1 + \alpha} \right) \tilde{v}) \) is non-empty, it must be the case that
\[
\beta_b v_j + (\kappa \left( \frac{1 + \alpha}{2} \right) v_j - 1 - \kappa) \eta_j^* = \beta_b \tilde{v},
\]
or
\[
(\kappa \left( \frac{1 + \alpha}{2} \right) v_j - 1 - \kappa) \beta_s (s_j^* - v_j) = \beta_s (1 + \kappa) (v_j - \tilde{v}).
\]
This simplifies, if \(\kappa \left( \frac{1 + \alpha}{2} \right) v_j - 1 - \kappa > 0\), which is \(\alpha < 1 + \frac{2}{\kappa}\), to
\[
s_j^* = \tilde{v} + \frac{\kappa \left( \frac{1 + \alpha}{2} \right) v_j - 1 - \kappa}{(\kappa \left( \frac{1 + \alpha}{2} \right) v_j - 1 - \kappa)} (v_j - \tilde{v}),
\]
which is the result. Note that this applies even when \(v_j = \left( \frac{1 + \kappa}{\kappa} \frac{2}{1 + \alpha} \tilde{v}\right)\), in which case \(s_j^* = 0\), and when \(v_j = \tilde{v}\), in which case \(s_j^* = v_j\), proving the proposition.

If \(\alpha = 1 + \frac{2}{\kappa}\), then \(\tilde{v} = \left( \frac{1 + \kappa}{\kappa} \frac{2}{1 + \alpha} \tilde{v}\right)\). In this case, for any \(v_j > \tilde{v}\), we must have \(s_j^* = 0\), and for any \(v_j < \tilde{v}\), we must have \(s_j = v_j\), which proves the live-or-die result.

G.5.3. \(\alpha > 1 + \frac{2}{\kappa}\). The case of selling everything is covered by \(\tilde{v} = v_N\). To prove that \(\tilde{v} > 0\), it suffices to note that the KKT conditions are necessary, and that the previous proof (the case of \(\alpha \in [-1, 1 + \frac{2}{\kappa}]\)) applies up to the definition of \(\tilde{v}\) (equation (G.4)). Because \(\alpha > -1\), we must
have $\bar{v} > 0$. Consider only the case in which the optimal security design is not selling everything or selling nothing going forward.

Note that monotonicity for the retained tranche requires that if $v_i = v_j$, $\eta^*_i = \eta^*_j$. Suppose that there exists an $i \in \Omega$, $i \in (0, N)$, with $v_N > v_i$, $\eta^*_{i-1} < \eta^*_i$ and $\eta^*_i \leq \eta^*_N < \beta_s v_N$. It is feasible to increase $\eta_N$ and decrease $\eta_i$, and therefore, applying lemma 4,

$$
(1 + \kappa - \frac{\kappa^{1+\alpha}}{2})(\eta^*_N - \eta^*_i) \geq \beta_s (v_N - v_i).
$$

This, however, is contradiction:

$$
(1 + \kappa - \frac{\kappa^{1+\alpha}}{2}) < 0.
$$

Therefore, if there exists an $i \in \Omega$, $i \in (0, N)$, with $\eta^*_i > \eta^*_{i-1}$ and $v_i < v_N$, we must have $\eta^*_N = \beta_s v_N$. Under the same circumstances, we can apply this logic to the second largest value in $v$ (so long as it is larger than $v_i$), noting that $\eta^*_N = \beta_s v_N$ ensures the monotonicity constraint does not bind. It follows that if there exists an $i \in \Omega$, $i > 0$, with $\eta^*_i > \eta^*_{i-1}$, then for all $j$ such that $v_j > v_i$, $\eta^*_j = \beta_s v_j$, and for all $j$ such that $v_j = v_i$, $\eta^*_j = \eta^*_j$.

Because $\eta^*_0 = 0$, there exists a $\tilde{i} \in \Omega$, such that, for all $i \leq \tilde{i}$, $\eta^*_i = 0$. By the assumption that the security is not selling everything, the largest possible index $\tilde{i}$ must satisfy $\tilde{i} < N$ and $\eta^*_{i+1} > 0$. It follows that for all $i > \tilde{i} + 1$ (if such indices exist), $\eta^*_i = \beta_s v_i$. Setting $\bar{v} = v_{i+1}$ proves the proposition.

**G.5.4. Proof that $\bar{v} < v_N$ if $v_N$ sufficiently large.** Suppose that $\bar{v} \geq v_N$, which implies the security is selling everything. In this case, the seller’s action in the moral hazard problem sets $p(0) = q$, and is therefore interior. It follows that, regardless of the value of $\alpha$, the solution to the moral hazard problem is interior, and therefore the security design equation described above must hold:

$$
\beta_b s_j(\eta^*) = \kappa \theta \left(\frac{p^0(\eta^*)}{q^0}\right)^{-\frac{1}{2}(1+\alpha)} \sum_{i > 0} [p^j(\eta^*) - \lambda^i + \omega^i] g_{ij}(p(\eta^*))
$$

$$
- \frac{\kappa}{2} \sum_{i > 0} [p^j(\eta^*) - \lambda^i + \omega^i] \delta_{ij} \frac{\eta^*_i}{p^j(\eta^*)}.
$$

Defining $\bar{v}$ as above,

$$
\bar{v} = E^{\eta^*}[s_j(\eta^*) + \beta_b^{-1}(\kappa \frac{1+\alpha}{2})(\eta^*_j + \frac{\omega^i v_i}{p^j(\eta^*)})] + \beta_b^{-1} \kappa \theta \left(\frac{p^0(\eta^*)}{q^0}\right)^{-\frac{1}{2}(1+\alpha)}.
$$

Specializing to selling everything,

$$
\bar{v} = E^q[v_j] + \beta_b^{-1} \kappa \theta.
$$
Our assumption was that

\[ v_N > \sum_i q^i v_i + \frac{\kappa}{\beta_b} \theta, \]

which implies \( v_N > \bar{v} \), a contradiction.

G.6. **Proof of corollary 3.** Recall the FOC for the security design problem:

\[ \frac{\partial U(\eta(\epsilon))}{\partial \epsilon} \bigg|_{\epsilon=0^+} = -\kappa \sum_{i \in \Omega} p^i(\eta^*) \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon=0^+} + \beta_b \sum_{i,j \in \Omega} s^*_j \frac{\partial p^i(\eta)}{\partial \eta_i} \bigg|_{\eta=\eta^*} \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon=0^+}. \]

It follows immediately that

\[ \kappa \sum_{i \in \Omega} p^i(\eta^*) \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon=0^+} = \kappa \frac{d}{d \epsilon} E^{p(\eta^*)}[\eta(\epsilon)]_{\epsilon=0^+} \]

\[ = -\kappa \frac{d}{d \epsilon} E^{p(\eta^*)}[\beta s(\epsilon)]_{\epsilon=0^+}. \]

As discussed in the text, \( \frac{d p^i(\eta)}{d \eta_i} \) is the inverse of Hessian matrix of the cost function. For the f-divergences,

\[ \frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} \bigg|_{p=p(\eta^*)} = \delta_{ij} \frac{\theta f''(\hat{p}^i(\eta^*))}{q^i} + \lambda_{ij} \frac{\theta f''(\hat{p}^i(\eta^*))}{q^0}. \]

Specializing this to the \( \alpha \)-divergences,

\[ f''(\frac{\hat{p}^i(\eta^*)}{q^i}) = \left(\frac{\hat{p}^i(\eta^*)}{q^i}\right)^{-\frac{1}{2}(3+\alpha)}. \]

It follows that

\[ \frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} \bigg|_{p=p(\eta^*)} = \delta_{ij} \frac{\theta}{(\hat{p}^i(\eta^*))^{-\frac{1}{2}(3+\alpha)} q^i} + \lambda_{ij} \frac{\theta}{(\hat{p}^i(\eta^*))^{-\frac{1}{2}(3+\alpha)} q^0}, \]

which, by the definitions of \( \hat{p} \) and \( \hat{\theta} \) (equation (4.2)), is

\[ \frac{\partial^2 \psi(p)}{\partial p^i \partial p^j} \bigg|_{p=p(\eta^*)} = \frac{\hat{\theta}(p(\eta^*)) \delta_{ij}}{\hat{p}^i(p(\eta^*))} + \frac{\hat{\theta}(p(\eta^*)) \lambda_{ij}}{\hat{p}^0(p(\eta^*))} = \hat{\theta}(p(\eta^*)) g_{ij}(\hat{p}(p(\eta^))). \]

Taking the inverse,

\[ \frac{\partial p^i(\eta)}{\partial \eta_i} \bigg|_{\eta=\eta^*} = \hat{\theta}(p(\eta^*))^{-1} g^{ij}(\hat{p}(p(\eta^*)) \hat{p}^j(p(\eta^*)). \]
where \( g^{ij} \) denotes the inverse Fisher information matrix. It follows that
\[
\beta_b \sum_{i,j \in \Omega} s_j^i \frac{\partial p^j(\eta)}{\partial \eta_i} \bigg|_{\eta = \eta^*} \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon = 0^+} = \beta_b \hat{\theta}(p(\eta^*))^{-1} E_{\hat{p}(p(\eta^*))} [s^* \cdot \frac{\partial \eta_i}{\partial \epsilon}]_{\epsilon = 0^+}
\]
\[- \beta_b \hat{\theta}(p(\eta^*))^{-1} E_{\hat{p}(p(\eta^*))} [s^*] E_{\hat{p}(p(\eta^*))} \frac{\partial \eta_i}{\partial \epsilon}]_{\epsilon = 0^+},
\]
which is
\[
\beta_b \sum_{i,j \in \Omega} s_j^i \frac{\partial p^j(\eta)}{\partial \eta_i} \bigg|_{\eta = \eta^*} \frac{\partial \eta_i}{\partial \epsilon} \bigg|_{\epsilon = 0^+} = -\beta_b \hat{\theta}(p(\eta^*))^{-1} \frac{d}{d\epsilon} V_{\hat{p}(p(\eta^*))} [s(\epsilon)]_{\epsilon = 0^+}.
\]
To finish the proof, I show that
\[
\frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{d e(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon = 0^+} = \hat{\theta}(p^*(\eta)) \frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{d}{\partial \epsilon} \text{Cov}_{\hat{p}(p(\eta^*))} [\eta(\epsilon), \beta_s v]_{\epsilon = 0^+},
\]
and that
\[
\frac{\beta_b}{\beta_s} \sum_{j \in \Omega} \frac{d p^j(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon = 0^+} (\eta^*_j - \gamma(\eta^*) \beta_s v_j) = \frac{1}{2} \hat{\theta}(p^*(\eta)) \frac{\beta_b}{\beta_s} \frac{\partial}{\partial \epsilon} \text{V}_{\hat{p}(p(\eta^*))} [\eta(\epsilon) - \gamma(\eta^*) \beta_s v]_{\epsilon = 0^+}.
\]
To prove the effort result, it is sufficient to show that
\[
\frac{d e(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon = 0^+} = \hat{\theta}(p^*(\eta))^{-1} \frac{\partial}{\partial \epsilon} \text{Cov}_{\hat{p}(p(\eta^*))} (\eta(\epsilon), \beta_s v)_{\epsilon = 0^+}
\]
\[- \hat{\theta}(p^*(\eta))^{-1} \sum_{i,j \in \Omega} \frac{\partial \eta^j}{\partial \epsilon} \bigg|_{\epsilon = 0^+} \beta_s v_i g^{ij}(\hat{p}(p(\eta^*))).
\]
The definition of effort is that
\[
\beta_s \sum_{i \in \Omega} (p^i(\eta^*) - q^i) v_i = e,
\]
and therefore
\[
\frac{d e(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon = 0^+} = \sum_{i \in \Omega} \frac{d p^i(\eta(\epsilon))}{d \eta_j} \bigg|_{\eta = \eta^*} \frac{\partial \eta^j}{\partial \epsilon} \bigg|_{\epsilon = 0^+} \beta_s v_i,
\]
which proves the result.
To prove the variance result, it is sufficient to show that
\[
\frac{\beta_b}{\beta_s} \sum_{j \in \Omega} \frac{d p^j(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon = 0^+} (\eta^*_j - \gamma(\eta^*) \beta_s v_j) = \frac{1}{2} \hat{\theta}(p^*(\eta))^{-1} \frac{\beta_b}{\beta_s} \frac{\partial}{\partial \epsilon} \text{V}_{\hat{p}(p(\eta^*))} [\eta(\epsilon) - \gamma(\eta^*) \beta_s v]_{\epsilon = 0^+}
\]
\[- \hat{\theta}(p^*(\eta))^{-1} \sum_{i,j \in \Omega} \frac{\partial \eta^j}{\partial \epsilon} \bigg|_{\epsilon = 0^+} (\eta^*_i - \gamma(\eta^*) \beta_s v_i) g^{ij}(\hat{p}(p(\eta^*)).
The result follows by the same logic:
\[ \hat{\theta} (p^*(\eta))^{-1} g^{ij} (\hat{\rho}(p(\eta^*))) \frac{\partial \eta^j}{\partial \epsilon} \bigg|_{\epsilon=0^+} = \frac{dp^j(\eta(\epsilon))}{d\epsilon} \bigg|_{\epsilon=0^+}. \]

G.7. Proof of lemma 2. The proof of this lemma uses Chentsov’s theorem and several results from Amari and Nagaoka [2007]. We have, for any monotone divergence,
\[ \frac{\partial^3 D(p||q)}{\partial p^i \partial p^j \partial p^k} \bigg|_{p=q} = c h_{ijk}(q) = c \partial_i g_{jk}(p) \bigg|_{p=q} + \Gamma_{jk,i}^{(\alpha)}, \]
where \( \Gamma_{jk,i}^{(\alpha)} \) are the connection coefficients of the \( \alpha \)-connection in the m-flat coordinate system. Using results in Amari and Nagaoka [2007], p. 33 and 36,
\[ \Gamma_{jk,i}^{(\alpha)} = \Gamma_{jk,i}^{(-1)} - \frac{1 + \alpha}{2} T_{ijk} = -\frac{1 + \alpha}{2} T_{ijk} \]
where \( T_{ijk} \) is a covariant symmetric tensor of degree three. Repeating the argument for the Riemannian connection,
\[ \Gamma_{ij,k} = -\frac{1}{2} T_{ijk} = \Gamma_{ik,j}^{(0)} \]
and
\[ c \partial_i g_{jk} = \Gamma_{ij,k}^{(0)} + \Gamma_{ik,j}^{(0)} = T_{ijk}. \]

It follows that
\[ h_{ijk}(q) = (\frac{3 + \alpha}{2}) \partial_i g_{jk}(p) \bigg|_{p=q}. \]

G.8. Additional Proposition.

Proposition 11. In the non-parametric model, with a smooth, convex, invariant divergence cost function, the difference in utilities achieved by an arbitrary security \( s \) and the sell-nothing security is, up to second order,
\[ U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = \kappa E_q^\beta [\beta_s s] - \frac{1}{2} \theta^{-1} V_q^\beta [\beta_s s] + \kappa \theta^{-1} \text{Cov}_q^\beta [\beta_s s, \eta(s)] - \frac{3 + \alpha}{12} \theta^{-2} K_3^q [\beta_s s, \beta_s s, \beta_s v] - \frac{3 + \alpha}{6} \theta^{-2} K_3^q [\beta_s s, \beta_s s, \beta_s s] + O(\theta^{-3} + \kappa \theta^{-2}). \]
To first order, the difference in utilities achieved by an arbitrary security \( s \) and the sell-nothing security is

\[
U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = \kappa E^q[\beta_s s_i] - \frac{1}{2} \theta^{-1} V^q[\beta_s s_i] + O(\theta^{-2} + \theta^{-1} \kappa).
\]

**Proof.** In this proof, I am using the summation convention. For example, \( p^i s_i \) is a summation, \( \sum_{i>0} p^i s_i \). We define the utility generated by a particular contract as

\[
U(s; \theta^{-1}) = \beta_s p^i (\eta(s)) s_i + \phi(\eta(s)).
\]

The first-order condition for the moral hazard problem is

\[
\eta_i - \partial_i \psi(p(\eta)) = 0.
\]

In the neighborhood of \( \theta \to \infty \),

\[
\lim_{\theta \to \infty} \partial_i D(p(\eta)||q) = 0,
\]

and therefore, by the strict convexity of \( D(\cdot||q) \),

\[
\lim_{\theta \to \infty} p(\eta) = q.
\]

The solution to the moral hazard problem is guaranteed to be interior in this neighborhood, due to the assumption that \( q \) has full support. Expanding the function \( \partial_i \psi(p(\eta)) \) around \( p = q \),

\[
\eta_i - \partial_i \psi(p(\eta)) - (p^j - q^j) \partial_i \partial_j \psi(q) - \frac{1}{2} (p^j - q^j)(p^k - q^k) \partial_i \partial_j \partial_k \psi(p^*) = 0,
\]

where \( p^* = q + a^*(p - q) \) for some \( a^* \in (0, 1) \). Because \( \psi(p) \) is invariant, we can simplify this to

\[
p^j - q^j = \theta^{-1} g^{ij}(q) \eta_i - \frac{1}{2} g^{ij}(q)(p^l - q^l)(p^k - q^k) h_{ikl}(p^*)
\]

where \( h_{ikl}(p^*) = \theta^{-1} c \partial_i \partial_j \partial_k \psi(p^*) \). It follows that

\[
p^j - q^j = O(\theta^{-1}).
\]

Returning to the first-order condition, and expanding it up to order \( \theta^{-2} \),

\[
\theta^{-1} \eta_i g^{il}(q) - \frac{1}{2} (p^j - q^j)(p^k - q^k) h_{ijk}(q) g^{il}(q) + O(\theta^{-3}) = (p^l - q^l).
\]

Plugging this equation into itself,

\[
\theta^{-1} \eta_i g^{il}(q) - \frac{1}{2} \theta^{-2} \eta_m \eta_n g^{mn}(q) g^{kn}(q) h_{ijk}(q) g^{il}(q) + O(\theta^{-3}) = (p^l - q^l).
\]
For ease of notation, define

\[ h^{lmn}(q) = g^{jm}(q)g^{kn}(q)h_{ijk}(q)g^{jl}(q). \]

It also follows that to second order, the cost function can be approximated as

\[
\psi(p) = \frac{\theta}{2}(p^i - q^i)(p^j - q^j)g_{ij}(q) + \frac{\theta}{6}(p^i - q^i)(p^j - q^j)(p^k - q^k)h_{ijk}(q) + O(\theta^{-3}),
\]

Using the first-order condition,

\[
\psi(p) = \frac{1}{2}(p^i - q^i)\eta_i - \frac{1}{4}\theta^{-1}(p^i - q^i)\eta_m\eta_n g^{jm}(q)g^{kn}(q)h_{ijk}(q) + \frac{\theta}{6}(p^i - q^i)(p^j - q^j)(p^k - q^k)h_{ijk}(q) + O(\theta^{-3}),
\]

and

\[
\psi(p) = \frac{1}{2}\theta^{-1}\eta_j\eta_i g^{ij}(q) - \frac{1}{4}\theta^{-2}\eta_m\eta_n\eta_i h^{im}(q) - \frac{1}{4}\theta^{-2}\eta_m\eta_n\eta_i h^{im}(q) + \frac{\theta}{6}\eta_m\eta_n\eta_i h^{im}(q) + O(\theta^{-3}).
\]

Simplifying,

\[
\psi(p) = \frac{1}{2}\theta^{-1}\eta_j\eta_i g^{ij}(q) - \frac{1}{3}\theta^{-2}\eta_m\eta_n\eta_i h^{im}(q) + O(\theta^{-3}).
\]

The utility given by an arbitrary security is

\[
U(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa)p^i(\eta(s))v_i - \kappa p^i(\eta(s))\eta_i(s) - \psi(p(\eta(s))),
\]

which under these approximations is
\[ U(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa)(p^i - q^i)v_i + \beta_s (1 + \kappa)q^i v_i - \kappa(p^i - q^i)\eta_i - \kappa q^i \eta_i - \frac{\theta}{2}(p^i - q^i)(p^j - q^j)g_{ij}(q) - \frac{\theta}{6}(p^i - q^i)(p^j - q^j)(p^k - q^k)h_{ijk}(q) + O(\theta^{-3} + \kappa \theta^{-2}). \]

and the expression can be rewritten as

\[
U(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa)\theta^{-1}\eta_i g^{il}(q)v_l - \frac{\theta^{-2}}{2} \beta_s \eta_j \eta_k v_l h^{ijk}(q) +
+ \beta_s (1 + \kappa)q^i v_i - \kappa \theta^{-1}\eta_i g^{il}(q)\eta_l - \kappa q^i \eta_i - \frac{\theta^{-1}}{2} \eta_i \eta_j g^{ij}(q) + \frac{\theta^{-2}}{3}\eta_i \eta_j \eta_k h^{ijk}(q) + O(\theta^{-3} + \kappa \theta^{-2}).
\]

(G.5)

For the zero security,

\[
U(0; \theta^{-1}, \kappa) = \frac{1}{2} \beta_s^2 \theta^{-1} v_i g^{il}(q)v_l - \frac{1}{6} \theta^{-2} \beta_s^3 v_j v_k v_l h^{ijk}(q) + \beta_s q^i v_i + O(\theta^{-3} + \kappa \theta^{-2}).
\]

Taking the difference,

\[
U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = \beta_s (1 + \kappa)\theta^{-1}\eta_i g^{il}(q)v_l - \frac{\theta^{-2}}{2} \beta_s \eta_j \eta_k v_l h^{ijk}(q) +
+ \beta_s \kappa q^i s_i - \frac{1}{2} \beta_s^2 \theta^{-1} v_i g^{il}(q)v_l + \frac{1}{6} \theta^{-2} \beta_s^3 v_j v_k v_l h^{ijk}(q) - \kappa \theta^{-1}\eta_i g^{il}(q)\eta_l - \frac{\theta^{-1}}{2} \eta_i \eta_j g^{ij}(q) + \frac{\theta^{-2}}{3}\eta_i \eta_j \eta_k h^{ijk}(q) + O(\theta^{-3} + \kappa \theta^{-2}).
\]

Substituting out (most) of the \( \eta \) terms for \( s \) and \( v \) terms,

\[
U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = -\frac{1}{2} \beta_s^2 \theta^{-1} s_l g^{il}(q)s_l + \beta_s \kappa \theta^{-1} \eta_i g^{il}(q)s_l + \kappa \beta_s q^i s_i
+ \frac{1}{6} \theta^{-2} \beta_s^2 \eta_j s_k v_l h^{ijk}(q) + \frac{1}{3} \theta^{-2} \beta_s^3 s_j s_k s_l h^{ijk}(q) + O(\theta^{-3}).
\]

This expression is equivalent to the statement of the theorem, except that it remains to show that

\[ s_i g^{il}(q)s_l = Var^9[s_i] \]
\[ v_i g^i(q) s_l = \text{Cov}^q[v_i, s_l], \]

and

\[ x_i y_j z_k h^{ijk}(q) = \left( \frac{3 + \alpha}{2} \right) K^q_3(x_j, y_k, z_i). \]

The first two follow from the definition of the non-parametric Fisher information metric (see also theorem 2.7 in Amari and Nagaoka [2007]). Applying lemma 2,

\[ x_i y_j z_k h^{ijk}(q) = \left( \frac{3 + \alpha}{2} \right) x_i g^i(q) y_j g^j(q) z_k g^k(p) \partial_l g_{mn}(p) \big|_{p=q}. \]

This can also be written in terms of the derivative of the inverse Fisher metric,

\[ x_i y_j z_k h^{ijk}(q) = -\left( \frac{3 + \alpha}{2} \right) x_i g^i(q) \partial_l (y_j z_k g^{jk}(p)) \big|_{p=q}. \]

Because of the non-parametric nature of the problem,

\[ y_j z_k g^{jk}(p) = E^p[y \cdot z] - E^p[y] E^p[z]. \]

Differentiating,

\[ \partial_l (y_j z_k g^{jk}(p)) \big|_{p=q} = y_l \cdot z_l - y_l E^q[z] - z_l E^q[y]. \]

Therefore,

\[ -\frac{2}{3 + \alpha} x_i y_j z_k h^{ijk}(q) = \left( \frac{3 + \alpha}{2} \right) x_i g^i(q) \partial_l (y_j z_k g^{jk}(p)) \big|_{p=q} \]

\[ = E^q[x \cdot y \cdot z] - E^q[x \cdot y] E^q[z] - E^q[x \cdot z] E^q[y] - E^q[y \cdot z] E^q[y] E^q[z] + 2 E^q[x] E^q[y] E^q[z], \]

which is the definition of the third cross-cumulant. \qed

G.9. **Proof of proposition 4.** This proof uses the summation convention described in the proof of proposition 11. From section G.8, the utility for a particular security design can be written as
Taking the FOC with respect to $\eta$, using $\lambda$ and $\omega$ as the multipliers on the limited liability constraints,

$$
\partial_i U(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) \theta^{-1} \eta_i g^i(q) v_i - \frac{\theta^{-2}}{2} \beta_s \eta_j \eta_k h^{ijk}(q) + \\
+ \beta_s (1 + \kappa) q^i v_i - \\
\kappa \theta^{-1} \eta_i g^i(q) \eta_l - \kappa q^i \eta_i - \\
\frac{\theta^{-1}}{2} \eta_i \eta_j g^{ij}(q) + \frac{\theta^{-2}}{3} \eta_i \eta_j \eta_k h^{ijk}(q) + O(\theta^{-3} + \kappa \theta^{-2}).
$$

where $\lambda^i$ and $\omega^i$ are the scaled limited liability multipliers. Recall the lemma,

$$
\eta_i g^i(q) + O(\theta^{-2}) = (p_l(q) - q_l),
$$

which proves the claims about $\tilde{\rho}$ and $p$. Note that

$$
\tilde{\rho}^i(p) = \frac{(p^i)^{(\alpha+3)}(q^i)^{-\frac{1}{2}(\alpha+1)}}{\sum_{j \in \Omega} (p^j)^{(\alpha+3)}(q^j)^{-\frac{1}{2}(\alpha+1)}},
$$

and therefore, to first order,

$$
\tilde{\rho}^i(p(\eta)) = \frac{1}{2}(\alpha + 3)(p^i(\eta) - q^i) = \tilde{\rho}(\eta).
$$

I will use $\tilde{\rho}$ and $\rho$ instead of $\tilde{\rho}(\eta)$ and $p(\eta)$ to keep the notation compact. Taylor expanding,
\[ g^{ij}(\bar{p}) = g^{ij}(q) - g^{ik}(q)(\partial_l g_{km}(p)|_{p=q})g^{mj}(q)(p^l - q^l) + O(\theta^{-2}) \]

Putting these two together,

\[ g^{ij}(\bar{p}) = g^{ij}(q) - \theta^{-1}h^{ijk}(q)\eta_k + O(\theta^{-2}). \]

Using these two results, the FOC can be written

\[
\partial^i U(s; \theta^{-1}, \kappa) = \beta_s \theta^{-1} g^{il}(\bar{p}) v_l + \\
\kappa \theta^{-1} g^{il}(q)(\beta_s s_l - \eta_l) - \kappa q^i - \\
\theta^{-1} \eta_j g^{ij}(\bar{p}) - \kappa (\lambda^i - \omega^i) + \\
O(\theta^{-3} + \kappa \theta^{-2}),
\]

which is equivalent to

\[
\partial^i U(s; \theta^{-1}, \kappa) = -\kappa p^i + \theta^{-1}(1 + \kappa) \beta_s s_j g^{ij}(\bar{p}) - \kappa (\lambda^i - \omega^i) + O(\theta^{-3} + \kappa \theta^{-2}).
\]

The proposition states that

\[
\frac{\partial U(\eta(\epsilon))}{\partial \epsilon} |_{\epsilon=0^+} = \frac{\partial}{\partial \epsilon} E^p[\beta_s s(\epsilon)] |_{\epsilon=0^+} - \frac{1}{2} \frac{\partial}{\partial \epsilon} V^{\eta(\epsilon)}[\beta_s s(\epsilon)] |_{\epsilon=0^+} + O(\theta^{-3} + \kappa \theta^{-2})
\]

Scaling the any feasible perturbation \( \frac{d\eta}{d\epsilon} \) proves the result.

The first-order result follows from

\[
\partial^i U(s; \theta^{-1}, \kappa) = -\kappa q^i + \theta^{-1} \beta_s s_j g^{ij}(q) - \kappa (\lambda^i - \omega^i) + O(\theta^{-2} + \kappa \theta^{-1}).
\]

G.10. **Proof of corollary 4.** The KKT conditions are necessary in the optimal security design problem. Therefore, we can use equation (G.6) and write

\[
0 = \beta_s \theta^{-1} g^{il}(\bar{p}) s_l + \\
\kappa \theta^{-1} g^{il}(q)(\beta_s s_l - \eta_l) - \kappa q^i - \\
\kappa (\lambda^i - \omega^i) + O(\theta^{-3} + \kappa \theta^{-2}).
\]
Using

$$\kappa(p^l - q^l) = \left(\frac{3 + \alpha}{2}\right)\kappa\theta^{-1}\eta_{il}g^il(q) + O(\kappa\theta^{-2}),$$

we have

$$\kappa(\lambda^i - \omega^j) = \beta_s\theta^{-1}g^il(\bar{p})s_l + \kappa\theta^{-1}g^il(q)\beta_s s_l - \kappa\bar{p}^i + \left(\alpha + \frac{1}{2}\right)\kappa\theta^{-1}\eta_{il}g^il(q) + O(\theta^{-3} + \kappa\theta^{-2}).$$

Note that

$$\kappa\theta^{-1}g^il(\bar{p}) = \kappa\theta^{-1}g^il(q) + O(\kappa\theta^{-2}).$$

Therefore,

$$\kappa(\lambda^i - \omega^j) = \beta_s\theta^{-1}(1 + \frac{1 - \alpha}{2})g^il(\bar{p})s_l - \kappa\bar{p}^i + \beta_s\left(\frac{1 + \alpha}{2}\right)\kappa\theta^{-1}v_l g^il(q) + O(\theta^{-3} + \kappa\theta^{-2}).$$

Solving for the security design,

$$s_l[\beta_s + \frac{1 - \alpha}{2}\beta_s\kappa]\theta^{-1} = \kappa(\bar{p}^i - \lambda^i + \omega^j)g^il(\bar{p}) - \frac{1 + \alpha}{2}\beta_s\kappa\theta^{-1}v_l + O(\theta^{-3} + \kappa\theta^{-2}).$$

This expression can be rewritten (ignoring higher order terms) as

$$\beta_s s_l(1 + \kappa) + \frac{1 + \alpha}{2}\kappa\eta_l(s) = \kappa\theta(\bar{p}^i - \lambda^i + \omega^j)g^il(\bar{p}).$$

Taking expected values under the $\bar{p}$ distribution,

$$\beta_s\bar{p}^i s_l(1 + \kappa) + \frac{1 + \alpha}{2}\kappa\bar{p}^i \eta_l(s) = \kappa\theta\bar{p}^i(\bar{p}^i - \lambda^i + \omega^j)g^il(\bar{p}).$$

Recalling that

$$g^il(\bar{p}) = \left(\frac{\delta_{il}}{\bar{p}^i} + \frac{1}{\bar{p}^i}\right),$$
\[ \beta_s \tilde{p}^l s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \tilde{p}^l \eta_l (s) = \kappa \theta \sum_{i} [\tilde{p}^i - \lambda^i + \omega^i] + \kappa \theta \sum_{i} [\tilde{p}^i - \lambda^i + \omega^i] \frac{1 - \tilde{p}^0}{\tilde{p}^l}. \]

Therefore,

\[ \beta_s \tilde{p}^l s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \tilde{p}^l \eta_l (s) = \kappa \theta \beta_s (1 + \kappa) \frac{\tilde{p}^0}{\tilde{p}^l}. \]

Plugging this back in,

\[ \beta_s s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \tilde{p}^l \eta_l (s) = \beta_s \tilde{p}^l s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \tilde{p}^l \eta_l (s) + \kappa \theta \sum_{i>0} [\tilde{p}^i - \lambda^i + \omega^i] \frac{\delta_{ij}}{\tilde{p}^l}. \]

Define

\[ \bar{v} = \tilde{p}^l s_l + \frac{1 + \alpha}{2} \kappa \tilde{p}^l \eta_l (s) + \kappa \theta \beta_s (1 + \kappa). \]

Suppose that \( \omega^i > 0 \), and therefore \( s_l = 0, \eta_l = \beta_s v_l, \lambda_l = 0 \). Then

\[ \frac{1 + \alpha}{2} \beta_s \kappa v_l = \beta_s (1 + \kappa) \bar{v} + \kappa \theta \omega^i. \]

This can never occur if \( \alpha \leq -1 \) and \( \bar{v} \geq 0 \), by the requirement that \( v_l > 0 \). If \( \alpha > -1 \), we know that \( \bar{v} > 0 \), and the condition occurs only if

\[ \frac{1 + \alpha}{2} \beta_s \kappa v_l - \beta_s (1 + \kappa) \bar{v} > 0, \]

which requires large values of \( v_l \). Next, consider the \( \lambda^i > 0 \) case, when \( s_l = v_l \) and \( \eta_l = 0 \) (and \( \omega_l = 0 \)). In this case,

\[ \beta_s v_l (1 + \kappa) = \beta_s (1 + \kappa) \bar{v} - \kappa \theta \frac{\lambda^i}{\tilde{p}^l}. \]

For any \( v_l < \bar{v} \), this condition can hold. Finally, consider the case when \( \lambda^i = \omega^i = 0 \). In this case,
\[ \beta_s s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \eta_l(s) = \beta_s (1 + \kappa) \bar{v}. \]

First, assume that \( \alpha \leq -1 \). In this case, it follows that if \( v_l < \bar{v} \), this condition cannot hold, and otherwise it can. Therefore, there are two regions for \( \alpha \leq -1 \): a region where \( s_l = v_l \), for \( v_l < \bar{v} \), and a region where

\[ s_l = \frac{(1 + \kappa)(1 + \alpha)}{(1 + \kappa - \frac{1 - \alpha}{2})} \bar{v} - \frac{1 + \alpha}{2} \kappa \frac{1 - \alpha}{2} v_l \]

for \( v_l > \bar{v} \). This simplifies to

\[ s_l = \bar{v} - \frac{\kappa(1 + \alpha)}{(2 + \kappa(1 - \alpha))} (v_l - \bar{v}). \]

It follows that

\[ \eta_l = -\frac{\beta_s (1 + \kappa)}{(1 + \kappa - \frac{1 - \alpha}{2})} \bar{v} + \frac{\beta_s (1 + \kappa)}{(1 + \kappa - \frac{1 - \alpha}{2})} v_l \]

in this region. The equation defining \( \bar{v} \) becomes

\[ \bar{v} \sum_{i : v_l < \bar{v}} \bar{p}^i = \sum_{i : v_l < \bar{v}} \bar{p}^i v_i + \frac{\kappa \theta}{\beta_s (1 + \kappa)} \]

which is guaranteed to be greater than zero. Next, consider \( \alpha \in (-1, 1 + \frac{2}{\kappa}) \). It follows that

\[ \frac{1 + \alpha}{2} \kappa \in (0, 1 + \kappa) \]

and therefore \( \bar{v} > 0 \). The condition that

\[ \frac{1 + \alpha}{2} \beta_s \kappa v_l - \beta_s (1 + \kappa) \bar{v} > 0 \]

is satisfied only for some \( v_l > v_{max} > \bar{v} \). Suppose that \( v_l > \bar{v} \). The left hand side of the zero-multiplier condition below,

\[ \beta_s s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \eta_l(s) - \beta_s (1 + \kappa) \bar{v} = 0, \]

achieves its minimum when \( \eta_l(s) = \beta_s v_l \), and therefore this cannot hold if \( v_l > v_{max} \). By a similar, argument, it achieves maximum when \( s_l = v_l \), and therefore cannot hold if \( v_l < \bar{v} \). It follows that there are three regions: a low \( v_l \) region, in which \( v_l < \bar{v} \), an intermediate \( v_l \in (\bar{v}, v_{max}) \) region in which

\[ s_l = \bar{v} - \frac{\kappa(1 + \alpha)}{(2 + \kappa(1 - \alpha))} (v_l - \bar{v}), \]
and a high $v_l$ region in which $s_l = 0$. Therefore, $s_{\text{debt-eq}}$ is the second-order optimal security design.

The statement that

$$U(s^*(\theta^{-1}, \kappa) ; \theta^{-1}, \kappa) - U(s_{\text{debt-eq}}(\theta^{-1}, \kappa) ; \theta^{-1}, \kappa) = O(\theta^{-3} + \kappa \theta^{-2})$$

follows, from the fact that $s_{\text{debt-eq}}(\theta^{-1}, \kappa)$ maximizes the non-$O(\theta^{-3} + \kappa \theta^{-2})$ terms of $U(s; \theta^{-1})$.

To first-order,

$$\kappa(\lambda^i - \omega^i) = \beta_s \theta^{-1} g^i(q) s_l - \kappa q^i + O(\theta^{-2} + \kappa \theta^{-1}).$$

This is exactly the equation above,

$$\beta_s s_l (1 + \kappa) + \frac{1 + \alpha}{2} \kappa \eta_l(s) = \kappa \theta (\bar{p}^i - \lambda^i + \omega^i) g_{i\theta}(\bar{p}),$$

specialized to the case of $\alpha = -1$, with $q$ instead of $\bar{p}$, and with a slightly different constant on $s$. As a result, the optimal security design in the case (debt) is first-order optimal, and the argument for first-order optimality follows.

G.11. **Proof of proposition 9.** This proof is similar in structure to proposition 11. I will continue to use the summation notation. We define the utility generated by a particular contract as

$$U(s; M_\xi, \theta^{-1}, \kappa) = \beta_s (1 + \kappa) p^i(\eta(s); M_\xi, \theta^{-1}) s_i + \phi(\eta(s); M_\xi, \theta^{-1}).$$

It will also be useful to define $p^i(\xi)$, which is the probability distribution induced by the parameters $\xi$. Define

$$B^i_a(\bar{p}) = \frac{\partial p^i(\xi)}{\partial \xi^a} |_{p = \bar{p}},$$

$$B^i_{ab}(\bar{p}) = \frac{\partial p^i(\xi)}{\partial \xi^a \partial \xi^b} |_{p = \bar{p}},$$

The first-order condition for the moral hazard problem is

$$B^i_a(\bar{p})(\eta_i - \partial_i \psi(p(\xi))) = 0.$$

As mentioned in the text, there may be multiple $\xi$ for which this FOC is satisfied. By assumption, however, the FOC holds at all $\xi$ that maximize the seller’s problem. In the neighborhood of $\theta \to \infty$, 

$$\lim_{\theta \to \infty} p(\eta; M_\xi, \theta^{-1}) = q,$$
because otherwise the seller would receive unbounded negative utility. By assumption, \( q \in M_\xi \), and therefore there are coordinates \( \xi_q \) such that \( p(\xi_q) = q \). Expanding the functions \( B_a^i(p(\xi)) \) and \( B_a^i(p(\xi)) \partial_i \psi(p(\xi)) \) around \( \xi = \xi_q \), using the fact that \( \partial_i \psi(q) = 0 \),

\[
B_a^i(p(\xi)) = B_a^i(q) + B_{ab}^i(q)(\xi^b - \xi_{q}^b) + B_{abc}^i(\check{p})(\xi^b - \xi_{q}^b)(\xi^c - \xi_{q}^c),
\]

where \( p^* = q + c^*(p - q) \) for some \( c^* \in (0, 1) \) and \( \check{p} \) similarly defined. Define \( g_{ab}(q) = B_a^i(q)B_b^i(q)g_{ij}(q) \) and \( g^{ab}(q) \) as its inverse. Define

\[
m_{ab}(q, \eta) = \theta g_{ab}(q) - B_{ab}^i(q)\eta_i
\]

and

\[
m_{abc}(\check{p}, p^*, \eta_i) = B_a^i(p^*)B_{bc}^i(p^*)\partial_i \partial_j \psi(p^*) + \frac{1}{2} B_a^i(p^*)B_{bc}^i(p^*)B_{jk}^i(p^*)\partial_i \partial_j \partial_k \psi(p^*) + \frac{1}{2} B_{abc}^j(p^*)\partial_i \partial_j \psi(p^*) - B_{abc}(\check{p})\eta_i.
\]

The FOC can be written, using the invariance property of \( f \)-divergences, as

\[
B_{a}^i(q)\eta_i = m_{ab}(q, \eta)(\xi^b - \xi_{q}^b) + m_{abc}(\check{p}, p^*, \eta_i)(\xi^b - \xi_{q}^b)(\xi^c - \xi_{q}^c).
\]

Multiplying by \( g^{ab}(q)\theta^{-1} \),

\[
\xi^b - \xi_{q}^b = \theta^{-1} g^{ab}(q)B_a^i(q)\eta_i + \theta^{-1}(\xi^c - \xi_{q}^c)g^{ab}(q)B_{ac}^i(q)\eta_i + \theta^{-1} g^{ab}(q)m_{acd}(\check{p}, p^*, \eta)(\xi^d - \xi_{q}^d)(\xi^c - \xi_{q}^c).
\]

Note that \( \theta^{-1}m_{acd}(\check{p}, p^*, \eta) = O(1) \) in \( \theta \). By the argument that \( \lim_{\theta \to \infty} p(\eta; M_\xi, \theta^{-1}) = q \), we know that \( \xi^b - \xi_{q}^b = o(1) \). It follows that

\[
\xi^b - \xi_{q}^b = \theta^{-1} g^{ab}(q)B_a^i(q)\eta_i + O(\theta^{-2}).
\]
This demonstrates that $\theta(\xi^b(\eta) - \xi^b)$ has a unique limit as $\theta \to \infty$. It follows that $p(\eta; M_\xi, \theta^{-1})$ is differentiable with respect to $\eta$ in the neighborhood of the expansion, regardless of the arbitrary rule used to “break ties” when multiple $p$ maximize the moral hazard sub-problem.

We now proceed as before, substituting in for the definition of $\beta_b$ and $\phi(\eta)$,

$$U(s; M_\xi, \theta^{-1}, \kappa) = \beta_s(1 + \kappa)p^i(\eta(s); M_\xi, \theta^{-1})v_i - \kappa p^i(\eta(s); M_\xi, \theta^{-1})\eta_i(s) - \psi(p(\eta(s); M_\xi, \theta^{-1})).$$

Expanding the function $p(\xi)$ around $\xi = \xi_q$, and using the result above,

$$p^i(\eta; M_\xi, \theta^{-1}) - q^i = \theta^{-1}B_a^i(q)g^{ab}(q)B_b^j(q)\eta_j + O(\theta^{-2}).$$

Expanding $\psi(p(\xi))$, again using the result from above,

$$\psi(p) = \frac{\theta^{-1}}{2}\eta_i\eta_j B_a^i(q)B_b^j(q)g_{ij}(q) + O(\theta^{-2}).$$

Plugging these two into the utility function,

$$U(s; M_\xi, \theta^{-1}, \kappa) = \beta_s(1 + \kappa)q^iv_i + \beta_s\theta^{-1}g^{ab}(q)B_a^i(q)B_b^j(q)\eta_i(s)v_j - \kappa q^i\eta_i(s) - \frac{1}{2}\theta^{-1}B_a^i(q)B_b^j(q)g^{ab}(q)\eta_i(s)\eta_j(s) + O(\theta^{-2} + \theta^{-1}\kappa).$$

The no-trade utility is

$$U(0; M_\xi, \theta^{-1}, \kappa) = \beta_s q^iv_i + \frac{1}{2}\beta_s^2\theta^{-1}B_a^i(q)B_b^j(q)g^{ab}(q)v_iv_j + O(\theta^{-2} + \theta^{-1}\kappa).$$

Taking the difference,

$$U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) = \beta_s\kappa q^iv_i + \beta_s\theta^{-1}B_a^i(q)B_b^j(q)g^{ab}(q)\eta_i(s)v_j - \frac{1}{2}\theta^{-1}B_a^i(q)B_b^j(q)g^{ab}(q)\eta_i(s)\eta_j(s) - \frac{1}{2}\beta_s^2\theta^{-1}B_a^i(q)B_b^j(q)g^{ab}(q)v_iv_j + O(\theta^{-2} + \theta^{-1}\kappa)$$

which simplifies to

$$U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) = \beta_s\kappa q^iv_i - \frac{1}{2}\beta_s^2\theta^{-1}B_a^i(q)B_b^j(q)g^{ab}(q)s_i\eta_j + O(\theta^{-2} + \theta^{-1}\kappa).$$

The lower bound,
\[ U(s; M_\xi, \theta^{-1}, \kappa) - U(0; M_\xi, \theta^{-1}, \kappa) \geq \kappa E^q[\beta_s s_i] - \frac{\theta^{-1}}{2} \text{Var}^q[\beta_s s_i] + O(\theta^{-2} + \theta^{-1} \kappa), \]

follows from theorems 2.7 and 2.8 of Amari and Nagaoka [2007]. Note that

\[ \frac{\partial}{\partial \xi} E^p[s_i]_{p=q} = s_i B^i_a(q). \]

From theorems 2.7 and 2.8, we have

\[ \text{Var}^q[s_i] = s_is_j g_{ij}(q) \geq s_i s_j B^i_a(q) B^j_b(q) g^{ab}(q). \]

The tightness of the lower bound applies when \( M_\xi \) is an exponential family, and \( s_i \) is a linear combination of its sufficient statistics, by the definition of a sufficient statistic.

**G.12. Proof of proposition 10.** This proof follows the proof of proposition 9. I will use the summation convention. We can write the utility from a particular security as

\[ U_{RI}(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) p^i(\eta(s); \theta^{-1}) s_i + \phi_{RI}(\eta(s); \theta^{-1}), \]

\[ \phi_{RI}(\eta; \theta^{-1}) = \max_{p\in M_{RI}} \{ p^i \eta_i - \theta D_{KL}(p||q(p)) \}. \]

Define a (for now) arbitrary distribution \( \bar{p} \), with the property that actions are independent of states, and the marginal distribution of states is \( g(x) \). We can write

\[ \psi(p) = \theta D_{KL}(p||q(p)) = \theta D_{KL}(p||\bar{p}) - \theta D_{KL}(q(p)||\bar{p}), \]

where \( q_a(p) \) refers to the marginal distribution over actions of \( q \), and \( \bar{p}_a \) is the marginal distribution of actions over \( \bar{p} \). The key to this idea is that \( q(p) \) has actions independent of states, and \( q_a(p) = p_a \), and therefore this equation holds for any full support \( \bar{p} \).

As noted earlier, the set of feasible probability distributions \( M_{RI} \) is an \( |X| \cdot (|A|-1) \) dimensional space, not an \( N \) dimensional space. It is an exponential family embedded in the space of all probability distributions. Denote a flat coordinate system \( \xi \), such that

\[ \frac{\partial p^i(\xi)}{\partial \xi^b} = B^i_b. \]

Note that unlike the general parametric model, \( B^i_b \) is a constant matrix. Similarly, define

\[ \frac{\partial q^i(p)}{\partial p^j} = C^i_j. \]

The first-order condition for the moral hazard problem is
\[ B^i_b(\eta_i - \partial_i \psi(p(\xi))) = 0. \]

As in the parametric problem, there may be multiple \( \xi \) for which this FOC is satisfied. By assumption, however, the FOC holds at all \( \xi \) that maximize the seller’s problem. In the neighborhood of \( \theta \to \infty \),

\[ \lim_{\theta \to \infty} p(\eta; \theta^{-1}) = \lim_{\theta \to \infty} q(p(\eta; \theta^{-1})), \]

because otherwise the seller would receive unbounded negative utility. This is different from the parametric problem, because even in the limit as \( \theta \) becomes large, the unconditional distribution of actions might depend on the retained tranche. The only requirement in this limit is that actions are independent of states. However, by the assumption of uniqueness, for each \( \eta \) there exists a unique \( \bar{p}(\eta) \) such that

\[ \lim_{\theta \to \infty} p(\eta; \theta^{-1}) = \bar{p}(\eta). \]

By assumption, \( \bar{p} \in M_\xi \), and therefore there are coordinates \( \xi_\bar{p} \) such that \( p(\xi_\bar{p}) = \bar{p} \). Expanding the function \( \partial_i \psi(p(\xi)) \) around \( \xi = \xi_\bar{p} \), using the fact that \( \partial_i \psi(\bar{p}) = 0 \),

\[ B^i_b \partial_i \psi(p(\xi)) = B^i_b B^j_c \partial_i \partial_j \psi(\bar{p}(\eta))(\xi^c - \xi^c_\bar{p}) + \frac{1}{2} B^i_d B^j_b B^k_c \partial_i \partial_j \partial_k \psi(p^*)(\xi^d - \xi^d_\bar{p})(\xi^c - \xi^c_\bar{p}), \]

where \( p^* = q + c^*(p - q) \) for some \( c^* \in (0, 1) \). Using the properties of the KL divergence,

\[ B^i_b B^j_c \partial_i \partial_j \psi(\bar{p}(\eta)) = \theta B^i_b B^j_c [g_{ij}(\bar{p}(\eta)) - C^d_i C^d_j g_{kl}(\bar{p}(\eta))] = \theta m_{bc}(\bar{p}(\eta)), \]

\[ B^i_b B^j_c B^k_d \partial_i \partial_j \partial_k \psi(p) = \theta B^i_b B^j_c B^k_d [h_{ijk}(p) - C^m_i C^m_j C^m_k g_{mn}(p)] = \theta m_{bcd}(p). \]

The FOC can be rearranged to

\[ m_{bc}(\bar{p}(\eta))(\xi^c - \xi^c_\bar{p}) = \theta^{-1} B^i_b \eta_i - \frac{1}{2}(\xi^d - \xi^d_\bar{p})(\xi^c - \xi^c_\bar{p})m_{bcd}(p^*). \]

Note that unlike the previous proof, \( m_{bc}(\bar{p}) \) is positive-semidefinite but singular, and therefore not invertible. Nevertheless, this is sufficient to show that

\[ m_{bc}(\bar{p}(\eta))(\xi^c - \xi^c_\bar{p}) = \theta^{-1} B^i_b \eta_i + O(\theta^{-2}). \]

We now proceed as before, substituting in for the definition of \( \beta_b \) and \( \phi(\eta) \),
\[ U(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) p^i(\eta(s); \theta^{-1}) v_i - \kappa p^i(\eta(s); \theta^{-1}) \eta_i(s) - \psi(p(\eta(s); \theta^{-1})). \]

Expanding the function \( p(\xi) \) around \( \xi = \xi_{\bar{p}} \), and using the result above,

\[ p^i(\eta; \theta^{-1}) - \bar{p}^i(\eta) = \theta^{-1} B^i_b(\xi^b - \xi^b_{\bar{p}}) + O(\theta^{-2}) \]

Expanding \( \psi(p(\xi)) \), again using the result from above,

\[ \psi(p) = \frac{\theta}{2} m_{bc} (\bar{p})(\xi^b - \xi^b_{\bar{p}})(\xi^c - \xi^c_{\bar{p}}) + O(\theta^{-2}). \]

Therefore,

\[ U(s; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) \bar{p}^i(\eta(s)) v_i + \beta_s B^i_b (\xi^b - \xi^b_{\bar{p}}) v_i - \kappa \bar{p}^i (\eta(s)) \eta_i(s) - \frac{\theta}{2} m_{bc} (\bar{p}(\eta(s))) (\xi^b - \xi^b_{\bar{p}})(\xi^c - \xi^c_{\bar{p}}) + O(\theta^{-2} + \theta^{-1} \kappa). \]

When the security is trading everything,

\[ U(v; \theta^{-1}, \kappa) = \beta_s (1 + \kappa) \bar{p}^i(0) v_i, \]

where \( \bar{p}^i(0) \) is the arbitrary decision the seller will make about unconditional actions, when they have no incentive whatsoever. When the security is trading nothing,

\[ U(0; \theta^{-1}, \kappa) = \beta_s \bar{p}^i (\beta_s v) v_i + \beta_s B^i_b (\xi^b_0 - \xi^b_{\bar{p}(0)}) v_i - \frac{\theta}{2} m_{bc} (\bar{p}(\beta_s v)) (\xi^b_0 - \xi^b_{\bar{p}(0)})(\xi^c_0 - \xi^c_{\bar{p}(0)}) + O(\theta^{-2} + \theta^{-1} \kappa), \]

where \( \xi^b_0 \) is the endogenous distribution for the zero security, and \( \xi^b_{\bar{p}(0)} \) is the associated limiting distribution. Because the expected payoff is the same for all actions, when those actions are independent of state,

\[ \bar{p}^i (\beta_s v) v_i = \bar{p}^i (\eta) v_i = \bar{p}^i (0) v_i, \]

and

\[ C^i_j v_i = 0 \]

for all \( j \). We know that for any \( \xi^b_{\text{ind}} \) that generates independence between actions and states,
\((\xi^{b}_{ind} - \xi^{b}_{p})m_{ab}(\bar{p})(\xi^{c}_{ind} - \xi^{c}_{\bar{p}}) = 0,\)

and therefore that \(\xi^{b}_{ind} - \xi^{b}_{p}\) is in the null space of \(m_{bc}(\bar{p})\). Therefore, the nullity of \(m_{bc}(\bar{p})\) is at least \(|A| - 1\). The rank of \(B^{i}_{b}B^{j}_{c}g_{ij}(\bar{p})\) is \((|A| - 1) \cdot |X|\) (full rank), and the rank of \(B^{i}_{b}B^{j}_{c}C^{k}_{i}C^{d}_{j}g_{kl}(\bar{p})\) is \(|A| - 1\), so the rank of \(m_{ab}(\bar{p})\) satisfies

\((|A| - 1) \cdot |X| \leq rank(m_{bc}(\bar{p}))+|A| - 1.\)

It follows that the nullity of \(m_{bc}(\bar{p})\) is exactly equal to \(|A| - 1\), the dimension of marginal distributions of actions. This fact is useful later in the proof. Moreover, because \(B^{i}_{b}C^{j}_{i}v_{i} = 0\), \(B^{i}_{b}v_{i}\) lies entirely in the column-space of \(m_{ab}(\bar{p})\). Therefore, there exists a vector \(v^{c}(\bar{p})\) such that

\[B^{i}_{b}v_{j} = m_{bc}(\bar{p})v^{c}(\bar{p}).\]

We can therefore rewrite

\[
U(s; \theta^{-1}, \kappa) - U(v; \theta^{-1}, \kappa) = \beta_{s}(\xi^{b} - \xi^{b}_{\bar{p}})m_{bc}(\bar{p}(\eta(s)))v^{c}(\bar{p}(\eta(s))) - \kappa \bar{p}^{i}(\eta(s))\eta_{i}(s) - \frac{\theta}{2}m_{bc}(\bar{p}(\eta(s)))((\xi^{b} - \xi^{b}_{\bar{p}})(\xi^{c} - \xi^{c}_{\bar{p}}) + O(\theta^{-2} + \theta^{-1}\kappa)).
\]

Consider a generalized inverse of \(m_{bc}(\bar{p})\), \(m^{bc}_{+}(\bar{p})\), which has the property that

\[m^{bc}_{+}(\bar{p})m_{bc}(\bar{p}) = m_{bc}(\bar{p}).\]

Rewriting the utility condition,

\[
U(s; \theta^{-1}, \kappa) - U(v; \theta^{-1}, \kappa) = \beta_{s}(\xi^{b} - \xi^{b}_{\bar{p}})m_{bc}(\bar{p}(\eta(s)))v^{c}(\bar{p}(\eta(s))) - \kappa \bar{p}^{i}(\eta(s))\eta_{i}(s) - \frac{\theta}{2}m_{bc}(\bar{p}(\eta(s)))m^{cd}_{+}(\bar{p}(\eta(s)))m_{bc}(\bar{p}(\eta(s)))((\xi^{b} - \xi^{b}_{\bar{p}})(\xi^{c} - \xi^{c}_{\bar{p}})) + O(\theta^{-2} + \theta^{-1}\kappa).
\]

Applying the first-order condition,

\[
U(s; \theta^{-1}, \kappa) - U(v; \theta^{-1}, \kappa) = \beta_{s}\theta^{-1}B^{i}_{b}B^{j}_{c}m^{bc}_{+}(\bar{p}(\eta(s)))\eta_{j}v_{j} - \kappa \bar{p}^{i}(\eta(s))\eta_{i}(s) - \frac{\theta^{-1}}{2}m^{bc}_{+}(\bar{p}(\eta(s)))B^{i}_{b}B^{j}_{c}\eta_{j}\eta_{j} + O(\theta^{-2} + \theta^{-1}\kappa).
\]

For the sell-nothing security,
\[ U(0; \theta^{-1}) - U(v; \theta^{-1}) = \frac{1}{2} \beta_s^2 \theta^{-1} B_b^i B_{e_i}^j m_{+}^{bc}(\bar{p}(\beta_s v))v_i v_j - \kappa \theta^{-1} \bar{p}^i(\beta_s v) v_i + O(\theta^{-2}). \]

Because

\[ g_{ab}(\bar{p}) = m_{ab}(\bar{p}) + B_b^i B_{e_i}^j C_i^k C_j^l g_{kl}(\bar{p}) \]

is non-singular, it follows from Hearon [1967] that a generalized inverse can be constructed as

\[ m_{+}^{bc}(\bar{p}) = g^{bd}(\bar{p}) m_{de}(\bar{p}) g^{ce}(\bar{p}) = g^{bc}(\bar{p}) - g^{bd}(\bar{p}) B_d^i B_{e_i}^j C_i^k C_j^l g_{kl}(\bar{p}) g^{ce}(\bar{p}). \]

Because \( B_b^i v_i \) is entirely in the column space of \( m_{bc}(\bar{p}) \), it is entirely in the row-space of \( m_{+}^{bc}(\bar{p}) \), and therefore

\[ m_{+}^{bc}(\bar{p}) B_b^i B_{e_i}^j v_i v_j = g^{bc}(\bar{p}) B_b^i B_{e_i}^j v_i v_j. \]

Because the unconditional variance of asset values is identical across assets,

\[ g^{bc}(\bar{p}(\beta_s v)) B_b^i B_{e_i}^j v_i v_j = g^{bc}(\bar{p}(\eta)) B_b^i B_{e_i}^j v_i v_j \]

for all \( \eta \). It therefore follows that

\[ U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) = \beta_s \kappa \bar{p}^i(\eta(s)) s_i - \frac{\theta^{-1}}{2} \beta_s^2 m_{+}^{bc}(\bar{p}(\eta(s))) B_b^i B_{e_i}^j s_i s_j + O(\theta^{-2} + \theta^{-1} \kappa). \]

By the positive-definiteness of \( g^{bd}(\bar{p}) B_d^i B_{e_i}^j C_i^k C_j^l g_{kl}(\bar{p}) g^{ce}(\bar{p}) \), and the monotonicity of the Fisher metric,

\[ U(s; \theta^{-1}, \kappa) - U(0; \theta^{-1}, \kappa) \geq \beta_s \kappa \bar{p}^i(\eta(s)) s_i - \frac{\theta^{-1}}{2} \beta_s^2 g^{ij}(\bar{p}(\eta(s))) s_i s_j + O(\theta^{-2} + \theta^{-1} \kappa), \]

which is the desired result.

G.13. **Additional Lemma.**

**Lemma 5.** For any effort strategy \( u(X, t) \in \mathcal{U} \), the stochastic exponential \( Z_t = \exp(\int_0^t u(X, s) dB_s - \frac{1}{2} \int_0^t u(X, s)^2 ds) \) is an everywhere-positive martingale, and the measure defined by \( \frac{dP}{dQ} = Z_1 \) is a measure in \( M \). Conversely, for any measure \( P \in M \), there exists an effort strategy \( u(X, t) \in \mathcal{U} \) such that \( \frac{dP}{dQ} = \exp(\int_0^1 u(X, s) dB_s - \frac{1}{2} \int_0^1 u(X, s)^2 ds) \). The effort strategy \( u(X, t) \) is unique up to an evanescence.
Proof. The first part of the lemma is a restatement of lemma 7.1 in Cvitanić et al. [2009]. The second part essentially restates a result found Bierkens and Kappen [2014], proposition 3.2. The uniqueness, up to an evanescence, of \( u(X,t) \) is shown below.

For all \( P \in M \),

\[
E^Q[(dP/dQ)^2] = E^P[dP/dQ] < \infty.
\]

By the inequality that \( \ln x < x - 1 \) for all \( x > 0 \), and the absolute continuity of \( dP/dQ \),

\[
E^P[\ln(dP/dQ)] + 1 < E^P[dP/dQ] < \infty.
\]

Therefore, \( D_{KL}(P||Q) \) is finite. It follows that the process \( u(X,t) \) given by proposition 3.2 of Bierkens and Kappen [2014] is square integrable, and therefore in the set \( \mathcal{U} \).

The semimartingale \( M_t \) that solves

\[
Z_t = E^Q[dP/F^B_t] = \exp(M_t - [M,M]_t)
\]
is unique, up to an evanescence (see chapter 2, theorem 8.3 in Jacod and Shiryaev [2003]). Therefore, it has a version such that

\[
M_t = \int_0^t u(X,s)dB_s,
\]

and is a square integrable martingale. By the Ito representation theorem, \( u(X,t) \) is the only square-integrable process for which this equation is satisfied.

G.14. Proof of proposition 5. This continuous time model described in the text is a version of the one discussed by Schaettler and Sung [1993] and Cvitanić et al. [2009], and uses the “strong” formulation of the moral hazard problem. There is a second approach to the moral hazard problem, known as the “weak” formulation, discussed by those authors. The weak formulation is closely related to the static problems discussed earlier, and is equivalent to the strong formulation for the purpose of determining the optimal security design.

In the weak formulation, let \( X_t \) be a stochastic process that evolves as

\[
dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t,
\]

where \( B \) is a Brownian motion on the probability space \( (\Omega, \mathcal{F}, Q) \), with standard augmented filtration \( \mathcal{F}^B_t \), and the \( b(\cdot) \) and \( \sigma(\cdot) \) are identical to the functions discussed in the text. Consider an \( \mathcal{F}^B_t \)-adapted control strategy \( u_t \) such that the stochastic exponential, \( Z_t = \exp(\int_0^t u_s dB_s - \frac{1}{2} \int_0^t u_s^2 ds) \), is a martingale. For such a strategy, Girsanov’s theorem holds, and we can define a measure,
$P$, that is absolutely continuous with respect to $Q$, such that $\frac{dP}{dQ} = Z_1$. Under the measure $P$, $B^P_t = B_t - \int_0^t u_s \, ds$ is a Brownian motion, and the process $X_t$ evolves as

$$dX_t = b(X_t, t) \, dt + u_t \sigma(X_t, t) \, dt + \sigma(X_t, t) \, dB^P_t.$$  

Under this measure $P$, the stochastic process $X$ has the same law as the asset price $V$ does under measure $\tilde{P}$ in the strong formulation of the problem. Because the assumed effort strategies are $\mathcal{F}^B_t$-adapted, they can be written as functionals $u_t = u(X, t)$. For any such control strategy, it follows that

$$\phi^{CT}(\eta; \{u_t\}) = E^P[\eta(X)] - E^P[\int_0^1 g(t, X_t, u(X, t)) \, dt],$$

meaning that the indirect utility in the strong formulation is the same as the indirect utility in the weak formulation. Here, I interpret $X$ as the asset price, and require that $\eta$ be $\mathcal{F}^X_t = \mathcal{F}^B_t$-measurable. By the same logic, $E^{\tilde{P}}[s(V)] = E^P[s(X)]$ under these effort strategies.

Defining the set $\mathcal{U}$ in this way ensures that Girsanov’s theorem can be applied. It follows that

$$\phi^{CT}(\eta) = \sup_{\{u_t\} \in \mathcal{U}} \{E^P[\eta(X)] - E^P[\int_0^1 g(t, X_t, u_t) \, dt]\},$$

meaning that the optimal strategies in the weak and strong formulations, assuming they exist and are unique, are identical.

Using lemma 5 above, we can note that there is a one-to-one relationship between strategies $\{u_t\} \in \mathcal{U}$ and measures in $M$. We can define a divergence,

$$D_g(P||Q) = \inf_{\{u_t\} \in \mathcal{U}} E^P[\int_0^1 g(t, X_t, u_t) \, dt],$$

subject to the constraint that $\frac{dP}{dQ} = \exp(\int_0^1 u_s dB_s - \frac{1}{2} \int_0^1 u^2_s ds)$. By the uniqueness result in lemma 5, all strategies $\{u_t\} \in \mathcal{U}$ that satisfy this constraint are identical for our purposes. Note that, because $g(t, X_t, u_t) = 0$ if and only if $u_t = 0$, and is otherwise positive, $D_g(P||Q)$ satisfies the definition of a divergence.

The moral hazard can be written as

$$\phi^{CT}(\eta) = \sup_{P \in M} \{E^Q[\frac{dP}{dQ} \eta(X)] - D_g(P||Q)\}.$$

Suppose there is a unique maximizer to this problem. The utility in the security design problem is

$$U(s) = \beta_b E^P[s(V)] + \phi^{CT}(\eta).$$

As noted above, $E^{\tilde{P}}[s(V)] = E^P[s(X)]$ and the result follows.

G.15. **Proof of proposition 6.** The relaxed moral hazard problem can be written as
\[ \phi_{CT}(\eta) = \sup_{dP} \{ E^P[\eta(X)] - \theta D_{KL}(P||Q) \}. \]

A suitably modified version of the proof of proposition 1 could be applied to this problem, but the extension of some of the shortcuts used in that proof is not straightforward. Instead, I will use a calculus-of-variations approach, similar to the earlier drafts of this paper. The result is a specialized version of Cvitanić et al. [2009]. I will assume that the space of allowed security designs is restricted to depend on the asset value at an arbitrary but finite set of times, which includes the final value. This assumption is necessary to use a theorem from the calculus of variations, but is not required for the proof of Cvitanić et al. [2009]. It also guarantees the existence of an optimal security design, by ensuring that the space of limited liability security designs is compact.

Let \( \hat{P} \) be an alternative measure on \( \Omega \) that is absolutely continuous with respect to \( Q \). Define

\[ \frac{dP(\eta, \alpha)}{dQ} = (1 - \alpha) \frac{dP^*(\eta)}{dQ} + \alpha \frac{\hat{P}}{dQ}, \]

where \( P^*(\eta) \) is the measure that maximizes the moral hazard problem. The retained tranche \( \eta : \Omega \to \mathbb{R} \) is a \( \mathcal{F}_t^B \)-measurable function on the sample space \( \Omega \). To keep notation compact, I use \( \eta \) instead of \( \eta(X) \) and \( \frac{dP}{dQ} \) instead of \( \frac{dP}{dQ}(B) \) when the meaning is not ambiguous.

\[ \phi_{CT}(\eta, \alpha) = \int_{\Omega} \frac{dP(\eta, \alpha)}{dQ} dQ - \theta \int_{\Omega} \frac{dP(\eta, \alpha)}{dQ} \ln(\frac{dP(\eta, \alpha)}{dQ}) dQ + \theta \int_{\Omega} \left[ \frac{dP(\eta, \alpha)}{dQ} - 1 \right] dQ, \]

we must have

\[ \left. \frac{\partial \phi_{CT}(\eta, \alpha)}{\partial \alpha} \right|_{\alpha=0} \leq 0 \]

for all \( \hat{P} \). It follows that, for all \( \hat{P} \),

\[ \int_{\Omega} \left[ \frac{d\hat{P}}{dQ} - \frac{dP^*(\eta)}{dQ} \right] [\eta - \theta \ln(\frac{dP^*(\eta)}{dQ})] dQ \leq 0. \]

This will be satisfied only for

\[ \frac{dP^*(\eta)}{dQ} = \exp(\theta^{-1}\eta - \lambda), \]

for some constant \( \lambda \). Of course, \( \frac{dP^*(\eta)}{dQ} \) must be a valid Radon-Nikodym derivative, and therefore

\[ \lambda = \ln(E^Q[\exp(\theta^{-1}\eta)]) > 0. \]
The integrability assumption is $E_Q[\exp(4^{-1}X_1)] < \infty$, and therefore $E_Q[\exp(\theta^{-1}\eta)]$ is finite for all limited liability $\eta$.

The security design problem can be written as

$$U_{CT}(\eta) = \beta_b E^P[X_1 - \beta_s^{-1}\eta] + \phi_{CT}(\eta)$$

$$= \beta_b E^P[X_1 - \beta_s^{-1}\eta] + \theta \ln(E_Q[\exp(\theta^{-1}\eta)]) .$$

This equation is identical to the one in Yang [2013] (proposition 3), and the proof (from this point) is essentially the same. I define

$$\eta(X, \epsilon) = \eta^*(X) + \epsilon \tau(X),$$

where $\tau : \Omega \rightarrow \mathbb{R}$ is another measurable function, restricted to the same set of payoff-relevant times. Using the calculus-of-variations approach again,

$$\frac{\partial U_{CT}(\eta(X, \epsilon))}{\partial \epsilon}|_{\epsilon=0} \leq 0$$

for all $\tau$ for which there exists some $\epsilon > 0$ such that $\eta(X, \epsilon)$ is a limited liability security. We have

$$\frac{\partial}{\partial \epsilon} \frac{dP^*(\eta^*)}{dQ}(B)|_{\epsilon=0} = \theta^{-1} \frac{dP^*(\eta^*)}{dQ}(B)(\tau(X(B)) - E^{P^*}[\tau]),$$

and

$$\frac{\partial}{\partial \epsilon} \phi(\eta)|_{\epsilon=0} = E^{P^*}[\tau].$$

It follows that, for all $\tau$, if an $\eta^*$ that maximizes $U_{CT}$ exists, then

$$(1 - \beta_b \beta_s) \int_{\Omega} \frac{dP^*(\eta^*)}{dQ} \tau dQ + \theta^{-1} \beta_b \int_{\Omega} \frac{dP^*(\eta^*)}{dQ}(\tau - E^{P^*}[\tau]) s dQ \leq 0.$$

This can be rearranged to

$$(G.7) \quad (1 - \beta_b \beta_s) \int_{\Omega} \frac{dP^*(\eta^*)}{dQ} \tau dQ + \theta^{-1} \beta_b \int_{\Omega} \frac{dP^*(\eta^*)}{dQ}(s - E^{P^*}[s]) dQ \leq 0.$$

Because the set of payoff-relevant times is fixed and finite, these integrals can be rewritten as integrals over $\mathbb{R}^M$, where $M$ is the number of payoff-relevant times. By the du Bois-Reymond lemma, for any interior $s(X)$,

$$\theta \beta_b - \beta_s \beta_b \beta s = s(X) - E^{P^*}[s(X)].$$
Note that if $s(X)$ is zero, decreasing $\eta(X)$ (increasing $s(X)$) will increase utility, and therefore $s(X) > 0$ for all paths with $X_1 > 0$. By an argument similar to proposition 1, the optimal contract is a debt security. The security depends only on the time-one value of the asset.

The decomposition of perturbation effects (equation (2.3)) into direct and indirect terms is exactly equation (G.7) above:

$$\frac{\partial U_{CT}(\eta(X, \epsilon))}{\partial \epsilon}\bigg|_{\epsilon=0} = \kappa \frac{\partial}{\partial \epsilon} E^{P^*(\eta^*)}[\beta_s s(X, \epsilon)] - (1 + \kappa) \frac{1}{2} \theta^{-1} \frac{\partial}{\partial \epsilon} V^{P^*(\eta^*)}[\beta_s s(X, \epsilon)].$$

In the continuous time setting, the decomposition of the indirect effect into effort and risk-shifting effects, defined in lemma 1, applies, modified to use integration instead of summation. The proof of the effort/risk-shifting decomposition follows along the lines of the proof of corollary 3. I will show that

$$\frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{d e(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon=0^+} = \theta^{-1} \frac{\beta_b}{\beta_s} (1 - \gamma(\eta^*)) \frac{\partial}{\partial \epsilon} Cov^{P^*(\eta^*)}[\eta(\epsilon), \beta_s X] \bigg|_{\epsilon=0^+},$$

and that

$$\frac{\beta_b}{\beta_s} d \int_{\Omega} (\eta^*(X) - \gamma(\eta^*) \beta_s X) d P(\eta(\epsilon)) = \frac{1}{2} \theta^{-1} \frac{\beta_b}{\beta_s} \frac{\partial}{\partial \epsilon} V^{P^*(\eta^*)}[\eta(\epsilon) - \gamma(\eta^*) \beta_s X] \bigg|_{\epsilon=0^+}.$$

To prove the effort result, it is sufficient to show that

$$\frac{d e(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon=0^+} = \theta^{-1} \frac{\partial}{\partial \epsilon} Cov^{P^*(\eta^*)}(\eta(\epsilon), \beta_s v) \bigg|_{\epsilon=0^+}.$$

The definition of effort is that

$$E^{P^*(\eta(\epsilon))}[\beta_s X] - E^{Q}[\beta_s X] = e,$$

and therefore

$$\frac{d e(\eta(\epsilon))}{d \epsilon} \bigg|_{\epsilon=0^+} = \frac{d}{d \epsilon} E^{P^*(\eta(\epsilon))}[\beta_s v] \bigg|_{\epsilon=0^+}$$

$$= \theta^{-1} \int_{\omega} \frac{d P^*(\eta^*)}{d Q}(B)(\tau(X(B)) - E^{P^*(\eta^*)}[\tau]) \beta_s X d Q,$$

which proves the result.

The variance result follows by the same logic:

$$\frac{\beta_b}{\beta_s} d \int_{\Omega} (\eta^*(X) - \gamma(\eta^*) \beta_s X) d P(\eta(\epsilon)) =$$

$$\frac{\beta_b}{\beta_s} \theta^{-1} \int_{\omega} \frac{d P^*(\eta^*)}{d Q}(B)(\tau(X(B)) - E^{P^*(\eta^*)}[\tau](\eta^*(X) - \gamma(\eta^*) \beta_s X) d Q,$$

which is the result.
G.16. **Proof of proposition 7.** I will proceed in four steps. In the first two steps, I will establish convergence results. In the third and fourth step, I will Taylor-expand the indirect utility function and buyer’s security valuation (the two components of the security design utility function). The extra steps in this proof, relative to the static models, arise because of the need to ensure integrability, and to prove that certain limits and integrals can be interchanged.

Define the retained tranche as a function of the \( Q \)-Brownian motion, \( \hat{\eta}(B) = \eta(X(B)) \). By limited liability, \( \hat{\eta}(B) \in [0, \beta^{-1} X(B)] \), and therefore \( E^P[\hat{\eta}(B)^2] < \infty \). It follows that \( \hat{\eta} \) is Hida-Malliavin differentiable (Di Nunno et al. [2008]). Define \( h_t = \int_0^1 u_s ds \). Following Monoyios [2013], the first-order condition for \( u_*^t \) to be optimal (assuming the bounds do not bind) is

\[
\psi'(u_*^t) = \theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B],
\]

where \( h^* = \int_0^1 u_*^s ds \). If the bounds do bind, so that \( |u_*| = \bar{u} \) at some time and state, then

\[
|\psi'(u_*^t)| \leq |\theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B]|
\]

By the mean value theorem,

\[
\psi''(\hat{u}_t^*) u_*^t = \theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B],
\]

for some \( |\hat{u}_t| \leq |u_*^t| \), if the bounds do not bind, and

\[
|\psi''(\hat{u}_t)| |u_*^t| \leq |\theta^{-1} E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B]|,
\]

if they do. Define \( e_t \) as

\[
e_t = E^Q[D_t \hat{\eta}(B) | \mathcal{F}_t^B].
\]

Additionally, define

\[
f_t = E^Q[D_t \hat{\eta}(B + h^*) | \mathcal{F}_t^B].
\]

Note that \( e_t \) does not depend on \( \theta \), whereas \( f_t \) depends on \( \theta \) through its dependence on \( h^* \).

The proof proceeds in several steps. First, I will show that \( u_*^t \) converges to zero, in the \( L^2(Q \times [0,1]) \) sense, meaning that

\[
\lim_{\theta^{-1} \to 0^+} E^Q[\int_0^1 (u_*^t)^2 dt] = 0.
\]

For all \( |u| \leq \bar{u}, \psi''(u) \geq K_1 > 0 \), and therefore

\[
(u_*^t)^2 \leq \theta^{-2} K_1^{-2} f_t^2,
\]
regardless of whether the bounds on \( u^*_t \) bind. By the assumption that \( \hat{\eta}(B + h^*) \) is in \( L^2(Q) \), for all \( h \), and the Clark-Ocone theorem for \( L^2(Q) \) (theorem 6.35 in Di Nunno et al. [2008]), \( f_t \in L^2(Q \times [0, 1]) \).

By the Itô isometry,

\[
E^Q[\int_0^1 f_t^2 dt] = E^Q[(\int_0^1 f_t dB_t)^2].
\]

By the Clark-Ocone theorem,

\[
\int_0^1 f_t dB_t = \eta(B + h^*) - E^Q[\eta(B + h^*)].
\]

Putting these two together,

\[
(G.8) \quad E^Q[\int_0^1 f_t^2 dt] = E^Q[(\eta(B + h^*) - E^Q[\eta(B + h^*)])^2].
\]

Because \( E^Q[(\eta(B + h)^2] < \infty \) for all feasible \( u \), and the set of feasible \( u \) is compact, it follows that

\[
\lim_{\theta^{-1} \to 0^+} E^Q[\int_0^1 f_t^2 dt] < \infty.
\]

Using this result, \( \lim_{\theta^{-1} \to 0^+} \theta^{-2} K_1^{-2} E^Q[\int_0^1 f_t^2 dt] = 0 \). By the squeeze theorem,

\[
\lim_{\theta^{-1} \to 0^+} E^Q[\int_0^1 (u^*_t)^2 dt] = 0.
\]

A similar application of the Itô isometry and Clark-Ocone theorem shows that

\[
(G.9) \quad E^Q[\int_0^1 e_t^2 dt] = E^Q[(\eta(B) - E^Q[\eta(B)])^2],
\]

which is useful later in the proof.

Next, I will show that \( \lim_{\theta^{-1} \to 0^+} \theta u^*_t = \lim_{\theta^{-1} \to 0^+} f_t = e_t \), with convergence in \( L^2(Q \times [0, 1]) \).

First, note that

\[
\lim_{\theta^{-1} \to 0^+} \theta u^*_t = \lim_{\theta^{-1} \to 0^+} \frac{1}{\psi'(\hat{u}_t)} f_t.
\]

Because \( |\hat{u}_t| \leq |u^*_t| \), it converges to zero in measure (\( L^2 \) convergence implies convergence in measure). Therefore, \( \lim_{\theta^{-1} \to 0^+} \frac{1}{\psi'(\hat{u}_t)} = 1 \), with convergence in measure. Because the measure \( Q \times \mu([0, 1]) \) is finite (\( \mu([0, 1]) \) is the Lebesgue measure), the product of two sequences that converge in measure converges in measure to the product of the limits. Therefore, to show that
\[ \lim_{\theta \to 0^+} \theta u_t^* = e_t, \text{ with convergence in measure, it is sufficient to show that } \lim_{\theta \to 0^+} f_t = e_t, \text{ with convergence in measure.} \]

Therefore, if

\[ \lim_{\theta \to 0^+} E^Q[\eta(B + h^*)^2] = E^Q[\eta(B)^2], \]

it would follow that \( f_t \) converges to \( e_t \) in the \( L^2(Q \times [0, 1]) \) sense.

Define

\[ Z_1 = \exp(\int_0^1 u_t^* dB_t - \frac{1}{2} \int_0^1 (u_t^*)^2 dt). \]

By Girsanov’s theorem,

\[ E^Q[\eta(B + h^*)^2] = E^Q[Z_1 \eta(B)^2] = E^Q[\eta(B)^2] + E^Q[(Z_1 - 1)\eta(B)^2]. \]

By the Cauchy-Schwarz inequality,

\[ E^Q[\eta(B + h^*)^2] = E^Q[\eta(B)^2] + E^Q[(Z_1 - 1)^2]^{0.5} E^Q[\eta(B)^4]^{0.5}. \]

By limited liability, \( E^Q[\eta(B)^4] \leq E^Q[X_1(B)^4] \), and by assumption is finite. By construction, \( E^Q[Z_1] = 1 \).

Therefore,

\[ E^Q[Z_1^2] = E^Q[\exp(2 \int_0^1 u_t^* dB_t - \int_0^1 (u_t^*)^2 dt)]. \]

Using the inequality that \((u_t^*)^2 \leq K_t^2 \theta^{-2} f_t^2\),
\[ E^Q[(Z_1 - 1)^2] \leq E^Q[\exp(2 \int_0^1 K_1^{-2} \theta^{-2} f_t^2 dt)] - 1 \]
\[ \leq E^Q[\exp(2 \int_0^1 K_1^{-1} \theta^{-1} f_t dB_t)] - 1 \]
\[ \leq E^Q[\exp(2K_1^{-1} \theta^{-1} (\eta(B + h^*) - E^Q[\eta(B + h^*)])] - 1 \]
\[ \leq E^Q[\exp(2K_1^{-1} \theta^{-1} X_1(B + h^*))] - 1 \]
\[ \leq E^Q[\exp(2K_1^{-1} \theta^{-1} X_1(B + \bar{u}))] - 1. \]

The first inequality follows from the inequality that \((u_t^*)^2 \leq K_1^{-2} \theta^{-2} f_t^2\). The second follows from the expectation of the stochastic exponential. The third applies the Clark-Ocone theorem. The fourth follows from limited liability. The fifth follows from the monotonicity of the asset value in effort.

By the assumption that, for all effort strategies, \(X_1\) is square-integrable, it follows that the moment-generating function exists in some neighborhood around zero. Therefore,

\[ \lim_{\theta^{-1} \to 0^+} E^Q[\exp(\sqrt{2} K_1^{-1} \theta^{-1} X_1(B + \bar{u}))] = 1. \]

It follows that

\[(G.10) \lim_{\theta^{-1} \to 0^+} E^Q[(Z_1 - 1)^2] = 0. \]

I have shown that \(f_t\) converges to \(e_t\), in the sense of \(L^2(Q \times [0, 1])\) convergence. It follows that \(f_t\) and \(\theta \bar{u}_t\) converge in measure to \(e_t\). Moreover, \(\theta^2(u_t^*)^2\) and \(f_t^2\) converge to \(e_t^2\) in measure, which will be useful below. Additionally, \(Z_1\) converges in measure to 1.

The third step is to Taylor-expand the indirect utility function from the moral hazard problem, in terms of \(\theta^{-1}\). To start, consider the first-order term:

\[ \frac{\partial}{\partial \theta^{-1}} \phi_{CT}(\eta; \theta) = \theta^2 E^P[\int_0^1 \psi(u_t^*) dt]. \]

By assumption, \(\forall |u| < \bar{u}, \psi''(u) \in [K_1, K_2]\) for some positive constants \(K_1\) and \(K_2\). Therefore, for some \(|\tilde{u}_t| < |u_t^*|\),
\[ \theta^2 \psi(u^*_t) = \frac{1}{2} \psi''(\tilde{u}_t) \theta^2 (u^*_t)^2 \]

\[ = \frac{1}{2} \psi''(\tilde{u}_t) f_t^2 \]

\[ \leq K f_t^2, \]

for some finite positive constant \( K = \frac{1}{2} \bar{K}^2 \).

Therefore,

\[ E^P \left[ \int_0^1 \theta^2 \psi(u^*_t) dt \right] \leq E^Q \left[ \int_0^1 K f_t^2 dt \right] + E^Q \left[ (Z_1 - 1) \int_0^1 K f_t^2 dt \right]. \]

To demonstrate that the second term on the right-hand side of this equation converges to zero, I prove the following lemma. The purpose of this lemma is only to establish that the second term converges to zero.

**Lemma 6.** With \( Z_1 - 1 \) and \( f_t \) defined as above,

\[ E^Q \left[ (Z_1 - 1) \int_0^1 f_t^2 dt \right] = \frac{1}{3} E^Q \left[ (Z_1 - 1)(\eta(B + h^*) - E^Q[\eta(B + h^*)])^2 \right]. \]

**Proof.** See appendix, section G.17. \( \square \)

Using this lemma and the Cauchy-Schwarz inequality,

\[ E^Q \left[ (Z_1 - 1)(\eta(B + h^*) - E^Q[\eta(B + h^*)])^2 \right] \leq E^Q \left[ (Z_1 - 1)^2 \right]^{0.5} E^Q \left[ (\eta(B + h^*) - E^Q[\eta(B + h^*)])^4 \right]^{0.5}. \]

Therefore, by the assumption that \( E^Q[\eta(B + h^*)] < \infty \), and equation (G.10),

\[ \lim_{\theta^{-1} \to 0^+} E^Q \left[ (Z_1 - 1) \int_0^1 f_t^2 dt \right] = 0. \]

Using equation (G.8) and Girsanov’s theorem,

\[ E^Q \left[ \int_0^1 f_t^2 dt \right] = E^Q \left[ (\hat{\eta}(B + h^*) - E^Q[\hat{\eta}(B + h^*)])^2 \right]. \]

\[ = E^Q \left[ Z_1 (\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2 \right] \]

\[ = E^Q \left[ (\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2 \right] + E^Q \left[ (Z_1 - 1)(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2 \right]. \]

By the Cauchy-Schwarz inequality and \( \lim_{\theta^{-1} \to 0^+} E^Q[\theta^{-1}] = 0 \),
\[
\lim_{\theta \to 0^+} E^Q[Z_1 \int_0^1 f_t^2 dt] = \lim_{\theta \to 0^+} E^Q[\int_0^1 f_t^2 dt] = E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])^2]
\]

The second step follows from equation (G.9).

Consider the sample space \( \Omega \times [0, 1] \), with the standard tensor-product sigma algebra and product measure \( Q \times \mu([0, 1]) \). The above result establishes convergence in \( L^1 \) of \( Z_1 f_t^2 \) to \( e_t^2 \), and therefore convergence in measure. By theorem 5 in chapter 11.6 of Shiryaev [1996], \( Z_1 f_t^2 \) is uniformly integrable. I will next argue that \( Z_1 \theta^2 \psi(u_t^*) \) is uniformly integrable.

By lemma 2 in chapter 11.6 of Shiryaev [1996], a necessary and sufficient condition for uniform integrability (which some authors use as the definition of uniform integrability) is that, for some random variable \( x_t \) that depends on \( \theta \), and for all \( \theta \),

\[
E^Q[\int_0^1 |x_t| dt] < \infty,
\]

and, for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that, for any set \( A \subseteq \Omega \times [0, 1] \) with \( E^Q[\int_0^1 1(A) dt] \leq \delta \), and all \( \theta \),

\[
E^Q[\int_0^1 |x_t| 1(A) dt] \leq \epsilon.
\]

By the inequality that \( \psi(u) \in [0, K f_t^2] \), it follows that

\[
\frac{1}{K} E^Q[\int_0^1 |Z_1 \theta^2 \psi(u_t^*)| dt] \leq E^Q[\int_0^1 |Z_1 f_t^2| dt] < \infty,
\]

and

\[
\frac{1}{K} E^Q[\int_0^1 |Z_1 \theta^2 \psi(u_t^*)| 1(A) dt] \leq E^Q[\int_0^1 |Z_1 f_t^2| 1(A) dt] \leq \epsilon.
\]

Therefore \( \frac{1}{K} Z_1 \theta^2 \psi(u_t^*) \) is also uniformly integrable. It follows that \( Z_1 \theta^2 \psi(u_t^*) \) is uniformly integrable. Taylor-expanding around \( u_t^* = 0 \), to second order,

\[
Z_1 \theta^2 \psi(u_t^*) = \frac{1}{2} Z_1 \psi''(\tilde{u}_t)(\theta u_t^*)^2,
\]

for some \( |\tilde{u}_t| \leq u_t^* \). By the convergence in measure of \( Z_1 \) to 1, \( u_t^* \) (and therefore \( \tilde{u}_t \)) to zero, and \( \theta u_t^* \) to \( e_t \), \( Z_1 \theta^2 \psi(u_t^*) \) converges in measure to \( \frac{1}{2} e_t^2 \). Because \( Z_1 \theta^2 \psi(u_t^*) \) converges in measure and
is uniformly integrable,

$$\lim_{\theta^{-1} \to 0^+} E^Q[Z_1 \int_0^1 \theta^2 \psi(u_t^*) dt] = E^Q[\int_0^1 e_t^2 dt] = \frac{1}{2} V^Q[\hat{\eta}(B)].$$

I have shown that the first-order term in the Taylor expansion converges to the variance. The zero-order term converges to the expected value of the retained tranche. Therefore, the Taylor expansion around the limit $\theta^{-1} \to 0^+$ is

$$\phi_{CT}(\eta; \theta) = E^Q[\hat{\eta}] + \theta^{-1} \frac{1}{2} V^Q[\hat{\eta}(B)] + O(\theta^{-1}).$$

The fourth step is to consider a Taylor expansion of the buyer’s security valuation, again around $\theta^{-1} \to 0^+$. Define $\hat{s}(B) = s(X(B))$. The buyer’s expected value is

$$E^Q[Z_1 \hat{s}(B)] = E^Q[\hat{s}(B)] + E^Q[(Z_1 - 1) \hat{s}(B)].$$

I will show that

$$\lim_{\theta^{-1} \to 0^+} E^Q[\theta(Z_1 - 1) \hat{s}(B)] = E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)]) \hat{s}(B)].$$

Using the stochastic logarithm,

$$Z_1 - 1 = \int_0^1 Z_t u_t^* dB_t,$$

where $Z_t u_t^*$ is in $L^2(\Omega \times [0, 1])$. By definition, $Z_t = E^Q[Z_1 | F_t]$. By equation (G.10), it follows that $Z_t$ converges in measure to 1, and therefore that $\theta Z_t u_t^*$ converges in measure to $e_t$.

Because $\hat{s}(B) \leq X_1(B)$ is in $L^2(\Omega)$, we can define

$$r_t = E^Q[D_t \hat{s}(B) | F^B_t]$$

and see that it is in $L^2(\Omega \times [0, 1])$. By the Clark-Ocone theorem,

$$E^Q[\theta(Z_1 - 1)(\hat{s}(B) - E^Q[\hat{s}(B)])] = E^Q[(\int_0^1 \theta Z_t u_t^* dB_t)(\int_0^1 r_t dB_t)].$$

Using the Ito isometry,

$$E^Q[(\int_0^1 \theta Z_t u_t^* dB_t)(\int_0^1 r_t dB_t)] = E^Q[\theta \int_0^1 Z_t u_t^* r_t dt].$$

By construction, $\theta Z_t u_t^*$ is square-integrable for all $\theta$. Because $\theta Z_t u_t^*$ converges in measure, it converges weakly on the space $L^2(\Omega \times [0, 1])$, which is a Hilbert space, and therefore
\[
\lim_{\theta^{-1} \to 0^+} E^Q[\theta \int_0^1 Z_t u_t^r r_t dt] = E^Q[\int_0^1 e_t r_t dt].
\]

Reversing the use of the Ito isometry and Clark-Ocone theorem,

\[
\lim_{\theta^{-1} \to 0^+} E^Q[\theta(Z_1 - 1)(\hat{s}(B) - E^Q[\hat{s}(B)])] = E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])(\hat{s}(B) - E^Q[\hat{s}(B)])].
\]

The term

\[
\lim_{\theta^{-1} \to 0^+} E^Q[\theta(Z_1 - 1)]E^Q[\hat{s}(B)] = \lim_{\theta^{-1} \to 0^+} E^Q[\int \theta Z_t u_t^r dB_t]E^Q[\hat{s}(B)] = 0.
\]

Therefore,

\[
\lim_{\theta^{-1} \to 0^+} E^Q[\theta(Z_1 - 1)\hat{s}(B)] = E^Q[(\hat{\eta}(B) - E^Q[\hat{\eta}(B)])\hat{s}(B)].
\]

The zero-order term in the buyer’s security valuation converges to

\[
\lim_{\theta^{-1} \to 0^+} E^Q[Z_1 \hat{s}(B)] = E^Q[\hat{s}(B)].
\]

The first-order term converges to

\[
\lim_{\theta^{-1} \to 0^+} \frac{\partial}{\partial (\theta^{-1})} E^Q[\frac{\partial Z_1}{\partial \theta} \hat{s}(B)] = \lim_{\theta^{-1} \to 0^+} E^Q[\theta(Z_1 - 1)\hat{s}(B)].
\]

Combining the results in the third and fourth steps, and also computing the first-order expansion of the buyer’s security valuation in terms of \(\kappa\), it follows that

\[
U(s; \theta^{-1}, \kappa) = (\beta_s + \kappa \beta_s)E^Q[\hat{s}(B)] + \theta^{-1} \beta_s E^Q[\hat{s}(B)\hat{\eta}(B)] - \\
\theta^{-1} \beta_s E^Q[\hat{s}(B)]E^Q[\hat{\eta}(B)] + E^Q[\hat{\eta}(B)] + \\
\frac{1}{2} \theta^{-1} V^Q[\hat{\eta}(B)] + O(\theta^{-2} + \theta^{-1} \kappa).
\]

This equation is equivalent to the first-order terms of equation (G.5) in appendix section G.8. The remainder of the proof of the mean-variance expression for the utility in the security design problem is identical to the algebra in that section.

The proof of section G.9 applies, ignoring the second order terms, and the effort and risk-shifting decomposition follows from the proof of proposition 6.
G.17. **Proof of lemma 6.** The purpose of this lemma is to derive an alternate form of a particular expression, which is shown to go to zero in the above proof. To accomplish this, I use a result regarding the relationship between Malliavin derivatives, Skorohod integrals, and cumulants, from Privault [2013]. I specialize that paper’s results to a simple case, involving adapted, square integrable processes that are themselves Malliavin derivatives. The proof relies heavily on the theorems of Malliavin calculus described in Di Nunno et al. [2008].

Using the martingale representation theorem, we can define

\[ Z_1 - 1 = \int_0^1 z_t dB_t, \]

where \( z_t \) is a square-integrable, \( \mathcal{F}_t^B \)-adapted process. Note that \( f_t \) also has these properties. It follows that both \( z_t \) and \( f_t \) are themselves Hida-Malliavin differentiable.

Using lemma 4.2 of Privault [2013],

\[
E_Q[(Z_1 - 1)(\int_0^1 f_t dB_t)] = E_Q[(Z_1 - 1)\int_0^1 f_t^2 dt] + E_Q[\int_0^1 \int_0^1 (D_t (Z_1 - 1))(D_s f_t) f_s ds dt] + E_Q[(\int_0^1 f_t dB_t) \int_0^1 f_t (D_t (Z_1 - 1)) dt].
\]

Using theorem 3.18 of Di Nunno et al. [2008], the fundamental theorem of Malliavin calculus,

\[
D_t (Z_1 - 1) = D_t (\int_0^1 z_r dB_r) = \int_0^1 (D_t z_r) dB_r + z_t.
\]

Therefore,

\[
E_Q[\int_0^1 \int_0^1 (D_t (Z_1 - 1))(D_s f_t) f_s ds dt] = E_Q[\int_0^1 \int_0^1 (\int_0^1 (D_t z_r) dB_r) (D_s f_t) f_s ds dt] + E_Q[\int_0^1 \int_0^1 z_t (D_s f_t) f_s ds dt].
\]
Analyzing the first term of this expression, using the integration by parts formula (theorem 6.15 of Di Nunno et al. [2008]), along with the Clark-Ocone theorem,

\[
E_Q \left[ \int_0^1 \left( \int_0^1 (D_t z_r) dB_r \right) f_t (\tilde{\eta}(B + h^*) - \tilde{\eta}) dt \right] = E_Q \left[ \int_0^1 \left( \int_0^1 (D_t z_r) dB_r \right) f_t \left( \tilde{\eta}(B + h^*) - \tilde{\eta} \right) dt \right] - \\
E_Q \left[ \int_0^1 \left( \int_0^1 (D_t z_r) dB_r \right) \int_0^1 f_t f_s dB_s dt \right].
\]

To minimize notation, I have used \( \tilde{\eta} = E_Q [\tilde{\eta}(B + h^*)] \) in the previous expression. Analyzing the first term of this expression,

\[
E_Q \left[ \int_0^1 \left( \int_0^1 (D_t z_r) dB_r \right) f_t (\tilde{\eta}(B + h^*) - \tilde{\eta}) dt \right] = E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) D_r (f_t (\tilde{\eta}(B + h^*) - \tilde{\eta})) dr dt \right]
\]

\[
= E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) f_t D_r (\tilde{\eta}(B + h^*) - \tilde{\eta}) dr dt \right]
\]

\[
= E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) f_t f_r dr dt \right] - \\
E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) f_t \left( \int D_r f_s dB_s \right) dr dt \right]
\]

\[
= E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) f_t f_r dr dt \right] - \\
E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) (D_r f_s) (D_s f_t) ds dr dt \right]
\]

\[
= E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) f_t f_r dr dt \right].
\]

The first step follows from integration by parts, the second by the fact (for adapted processes \( z_r \) and \( f_t \)) that \( (D_t z_r)(D_r f_t) \) integrates to zero (see remark 6.18 of Di Nunno et al. [2008]). The third step follows from the fundamental theorem, and the fourth from integration by parts. The fifth step applies the same step about adapted processes.

By the Ito isometry,

\[
E_Q \left[ \int_0^1 \left( \int_0^1 (D_t z_r) dB_r \right) \int_0^1 f_t f_s dB_s dt \right] = E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) f_t f_r dr dt \right].
\]

Putting these last two results together,

\[
E_Q \left[ \int_0^1 \int_0^1 (D_t z_r) dB_r (D_s f_t) f_s ds dt \right] = 0.
\]

Next, consider the term
\[ E^Q[(\int_0^1 f_t dB_t)^2] = E^Q[(\int_0^1 f_t dB_t)^2] + E^Q[\int_0^1 z_t (D_t f_t) f_t dr dt] + E^Q[\int_0^1 (D_t z_t) f_t f_t dr dt] + E^Q[\int_0^1 (\hat{\eta}(B + h^*) - \bar{\eta}) f_t z_t dt]. \]

Using the chain rule of Malliavin calculus,

\[ E^Q[(Z_1 - 1)(\int_0^1 f_t dB_t)^2] = E^Q[(Z_1 - 1)(\int_0^1 f_t^2 dt] + E^Q[\int_0^1 (D_t z_t) f_t f_t dr dt] + E^Q[(\hat{\eta}(B + h^*) - \bar{\eta}) f_t z_t dt]. \]

Integrating by parts,

\[ E^Q[(Z_1 - 1)(\int_0^1 f_t dB_t)^2] = E[(Z_1 - 1)(\int_0^1 f_t^2 dt] + 2E^Q[\int_0^1 (\hat{\eta}(B + h^*) - \bar{\eta}) f_t z_t dt]. \]

Finally, I figure out what the second term in this expression is.
The first step uses integration by parts and the Clark-Ocone theorem. The second uses the chain rule, Clark-Ocone theorem, and then fundamental theorem. Considering the second term,

\[ E_Q[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 z_t(\int_0^1 (D_t f_s) dB_s) dt] = E_Q[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 z_t (\int_0^1 (D_t f_s) dB_s) dt] + E_Q[(\eta(B + h^*) - \bar{\eta}) \int_0^1 z_t (\int_0^1 (D_t f_s) dB_s) dt]. \]

The first step is integration by parts, and the second applies the previously mentioned fact about adapted processes. The third uses the fundamental theorem, and the fourth uses both the chain rule (in the first term) and integration by parts (in the second term). The last step uses the fact about adapted processes and integration by parts, along with the Clark-Ocone theorem. It follows that

\[ E_Q[(\hat{\eta}(B + h^*) - \bar{\eta}) \int_0^1 z_t^2(\int_0^1 f_s ds)] = E_Q[(\hat{\eta}(B) - \bar{\eta}) \int_0^1 f_t z_t dt]. \]

Plugging in this into the earlier equation,

\[ E_Q[(Z_1 - 1)(\int_0^1 f_t dB_t)^2] = E[(Z_1 - 1) \int_0^1 f_t^2 dt] + \frac{2}{3} E_Q[(\hat{\eta}(B + h^*) - \bar{\eta})^2(Z_1 - 1)]. \]

Using the Clark-Ocone theorem,
\[ E^Q[(\hat{\eta}(B + h^*) - \hat{\eta})^2(Z_1 - 1)] = 3E^Q[(Z_1 - 1) \int_0^1 f'_t \, dt], \]

which proves the lemma.

G.18. **Proof of proposition 8.** For the definitions of the strong proper equilibrium concept, see definitions 1.2 and 3.2 of Simon and Stinchcombe [1995]. Let \( s^*, p^*, \) and \( k^* \) be the unique optimal security design, actions, and price in the principal-agent timing. First, note that the price \( k^* \) gives the buyer a strict incentive to accept: \( k^* < \beta \sum_{i>0} p^* s_i^* \). Otherwise, if equality held, the principal-agent equilibrium would not be unique; the buyer could accept with a high probability less than one, such that the seller was still better off offering \( s^* \) and \( k^* \) than deviating to some alternative.

Consider the seller’s mixed strategy, \( \mu(p, s, k) \), where \( \mu \) is a measure over \( M \times S \times K \). Hold fixed the buyer’s strategy: let \( a(s, k) \) denote the probability of accepting given security \( s \) and price \( k \). Define the seller’s utility as

\[
U_s(p, s, k; a) = \beta_s p^i v_i - \psi(p) + a(s, k)(k - \beta_s p^i s_i),
\]

where I have used the summation notation \( p^i s_i = \sum_i p^i s_i \). Note that this is different from the principal-agent utility, because the actions are taken before the security is traded. By assumption, \( \psi(p) \) is strictly convex in \( p \), and therefore there is a unique maximizer

\[
p(s, k; a) = \arg \max_{p \in M} U_s(p, s, k; a).
\]

For each \( s \) and \( k \), given the buyer’s strategy \( a(s, k) \), partition \( M \) into

\[
M^+(s, k; a) := \{ p \in M | U(p(s, k; a), s, k; a) - U(p, s, k; a) < \varepsilon^2 \},
\]

for \( \varepsilon > 0 \) and \( M^-(s, k; a) = M \setminus M^+(s, k; a) \). By the finiteness of \( S \) and \( K \), this is a finite measurable partition. To construct an \( \varepsilon \)-proper equilibrium, it must follow that

\[
\mu(M^-(s, k; a), s, k) \leq \varepsilon \mu(M^+(s, k; a), s, k).
\]

By the Bayes’ rule,

\[
\int_{p \in M^+} d\mu(p|s, k) = \frac{\mu(M^+|s, k)}{\mu(M^-(s, k; a), s, k) + \mu(M^+(s, k; a), s, k)} \geq \frac{1}{1 + \varepsilon}.
\]
Now consider the buyer’s strategy. The buyer’s utility from acceptance, conditional on the seller’s strategy, is

\[ U_b(s, k; \mu) = \beta_b \int_{p \in M} p^i s_i d\mu(p|s, k) - k. \]

By the limited liability of the security design, this can be rewritten to an inequality,

\[ U_b(s, k; \mu) \geq \beta_b \int_{p \in M^+} p^i s_i d\mu(p|s, k) - k. \]

By the strict convexity of \( \psi(p) \) and the compactness of \( M \), the seller’s utility \( U_s(p, s, k; a) \) is strongly concave in \( p \). We can write

\[ U(p(s, k; a), s, k; a) - U(p(s, k; a)) \geq \frac{m}{2} \| p - p(s, k; a) \|_2^2, \]

for some constant \( m > 0 \). It therefore follows that for all \( i \),

\[ |p^i - p^i(s, k; a)| \leq \varepsilon \sqrt{\frac{2}{m}}. \]

By limited liability, \( s_i \leq v_N \) for all \( i \), and therefore

\[ p^i s_i \geq p^i(s, k; a)s_i - \varepsilon v_N \sqrt{\frac{2}{m}}. \]

Therefore,

\[ U_b(s, k; \mu) \geq \frac{\beta_b}{1 + \varepsilon}(p^i(s, k; a)s_i - \varepsilon v_N \sqrt{\frac{2}{m}}) - k. \]

For any price, security design, and acceptance strategy such that \( \beta_b p^i(s, k; a)s_i - k > 0 \), it follows that there exists a \( \varepsilon(s, k, a) \) such that, for all \( \varepsilon \leq \varepsilon(s, k, a) \), accepting the offer strictly dominates rejecting the offer.

Next, I will show that the bound can be made uniform in acceptance strategies. The first-order condition that defines \( p(s, k; a) \) is

\[ \beta_s(v_i - a(s, k)s_i) = \partial_i \psi(p(s, k; a)). \]

Differentiating with respect to \( a(s, k) \),
\[-\beta_s s_i = \partial_j \partial_t \psi(p(s, k; a)) \frac{\partial p^i(s, k; a)}{\partial a(s, k)}.
\]

It follows that

\[-\beta_s \frac{\partial p^i(s, k; a)}{\partial a(s, k)} s_i > 0,
\]

by the strict convexity of \(\psi(p)\). Therefore, for all \(a\),

\[p^i(s, k; a) s_i \geq p^i(s, k; 1) s_i,
\]

where the latter denotes acceptance with certainty. Therefore, for any \(s, k\) with \(\beta_b p^i(s, k; 1) s_i - k > 0\), there exists an \(\varepsilon(s, k)\), such that, for all \(\varepsilon < \varepsilon(s, k)\), acceptance strictly dominates rejection. It follows from the definition of an \(\varepsilon\)-proper equilibrium that \(a(s, k) \geq \frac{1}{1+\varepsilon}\), for all such securities and prices and \(\varepsilon < \varepsilon(s, k)\). By the finiteness of \(S\) and \(K\), this bound can be made uniform: there exists a \(\varepsilon\) such that, for all \(\varepsilon < \varepsilon\), \(a(s, k) \geq \frac{1}{1+\varepsilon}\) for all \(s, k\) such that \(\beta_b p^i(s, k; 1) s_i - k > 0\).

Returning to the seller’s utility function, let \(s^*\) and \(k^*\) be the optimal security design and price in the principal-agent timing. As discussed above,

\[\beta_b p^i(s^*, k^*; 1) s^*_i - k^* > 0.
\]

It follows that, in any strong (or weak) proper equilibrium,

\[a^*(s^*, k^*) = 1,
\]

meaning that the equilibrium security and price from the principal-agent timing game will be accepted if offered.

Now consider a Nash equilibrium that involves the play of some security \(\hat{s}\) and some price \(\hat{k}\), and satisfies \(a^*(s^*, k^*) = 1\). I consider two cases: \(\hat{s}, \hat{k}\) that are rejected with certainty, and \(\hat{s}, \hat{k}\) that are accepted with positive probability. The play of any \(\hat{s}, \hat{k}\) that are rejected with certainty cannot be an equilibrium, because the seller could deviate to \(s^*, k^*, p^*\) and achieve higher utility (otherwise, selling nothing would be an equilibrium in the principal-agent timing).

Now consider an Nash equilibrium in which \(\hat{s}\) and \(\hat{k}\) are accepted with positive probability. Let \(\hat{p}\) denote the an action played in this equilibrium. It must be case that

\[\hat{p} = p(\hat{s}, \hat{k}; a^*),
\]

because otherwise the seller could deviate and achieve higher utility. Now consider the buyer. For the buyer to accept with positive probability, we must have
\[ \beta b \hat{p}_i \hat{s}_i - \hat{k}_i \geq 0. \]

The seller’s utility is

\[ U_s(\hat{p}, \hat{s}, \hat{k}; a^*) = \beta s \hat{p}_i v_i - \psi(\hat{p}) + a^*(\hat{s}, \hat{k})(k - \beta s \hat{p}_i \hat{s}_i). \]

Assume that \( k - \beta s \hat{p}_i \hat{s}_i \geq 0 \). In that case,

\[ U_s(\hat{p}, \hat{s}, \hat{k}; a^*) \leq \beta s \hat{p}_i v_i - \psi(\hat{p}) + \beta b \hat{p}_i \hat{s}_i - \beta s \hat{p}_i \hat{s}_i, \]

and therefore \( U_s(\hat{p}, \hat{s}, \hat{k}; a^*) \leq U_s(p^*, s^*, k^*; a^*), \) strictly if \( \hat{p} \neq p^*, \hat{s} \neq s^*, \) or \( \hat{k} \neq k^* \) by the uniqueness of the equilibrium in the principal-agent timing.

Now assume that \( k - \beta s \hat{p}_i \hat{s}_i < 0 \). In the case,

\[ U_s(\hat{p}, \hat{s}, \hat{k}; a^*) \leq \beta s \hat{p}_i v_i - \psi(\hat{p}), \]

and is therefore weakly dominated by selling nothing, which is strictly dominated by \( p^*, s^*, k^* \).

It follows that all Nash equilibria in which \( a^*(s^*, k^*) = 1 \) are characterized by the play of \( p^*, s^*, k^* \). Theorem 3.1 of Simon and Stinchcombe [1995] shows that a strong proper equilibrium exists, and I have shown that all strong proper equilibria are the game are characterized by \( a^*(s^*, k^*) = 1 \). Because strong proper equilibria are Nash equilibria, it follows that all strong proper equilibria are characterized by the play of \( p^*, s^*, k^* \), and acceptance by the buyer.